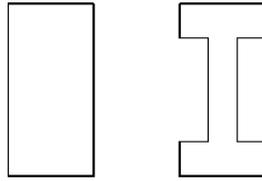


An Example of Shape Consideration for Elastic Deformation

As a reference point, we start with a steel beam with a rectangular cross-section (Case 1). We will then compare strategies for reducing the weight, requiring that the elastic deflection remain the same: using an aluminum beam with a rectangular cross-section (Case 2); using a steel I-beam instead of a rectangular shape (Case 3); and using an aluminum I-beam (Case 4).

The formulae cited here come from a field of study often referred to as strength of materials. A main objective is to calculate the deflection of beams subjected to various loads.



For material properties, we will use:

Steel $\rho = 7900 \text{ kg/m}^3$ (density)
 $E = 200 \text{ GPa}$ (elastic modulus)

Aluminum $\rho = 2700 \text{ kg/m}^3$
 $E = 70 \text{ GPa}$

For geometry, we will use a base width of $b = 4.0 \text{ cm}$ for both beams, for all cases. For I-beams, we will use 1.0 cm for both the flange and web thicknesses. This I-beam with square corners is stylized for ease of computation. For actual I-beams with tapered flanges and rounded corners, the properties of standard beams are tabulated, or we use computer software known alternatively as *finite element analysis* (FEA) or the *finite element method* (FEM).

Case (1): Look at the images above. The beam on the left is made of steel. Consider that the height of the beam is $h_1 = 6.0 \text{ cm}$. Then, the mass per unit length of the beam is given by:

$$\begin{aligned} m_{L1} &= \rho A \\ A &= b h_1 \end{aligned} \quad \text{(cross-sectional area)}$$

$$\begin{aligned} A_1 &= (4.0)(6.0) \\ &= 24.0 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} m_{L1} &= (7900)(24.0 \times 10^{-4}) \\ &= 19.0 \text{ kg/m} \end{aligned}$$

Case (2): Look at the images above. The beam on the left is made of aluminum.

From strength of materials books, elastic beam deflection is inversely proportional to the product EI . The term I is a geometrical property of the cross-sectional shape. It is somewhat loosely called the moment of inertia. The more correct name is moment of area, since I has units of m^4 , not the $kg \cdot m^2$ of the rotational moment of inertia. Using subscripts 2 for this case and subscripts 1 for the previous, we now require that:

$$\begin{aligned} I_2 &= (E_1/E_2) I_1 \\ &= (200/70) I_1 \\ &= 2.86 I_1 \end{aligned}$$

We do not have to calculate the moment of inertia here. Simply know that for a rectangular shape, the formula is:

$$I = bh^3/12$$

So, the height of an aluminum beam having the same elastic deflection as the Case (1) steel beam is computed to be:

$$\begin{aligned} h_2 &= (2.86)^{1/3}(6.0) \\ h_2 &= 8.51 \text{ cm} \end{aligned}$$

$$\begin{aligned} A_2 &= (4.0)(8.51) \\ &= 34.0 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} m_{L2} &= (2700)(34.0 \times 10^{-4}) \\ &= 9.2 \text{ kg/m} \end{aligned}$$

This beam is nearly 50% lighter than the steel beam of Case (1). However, an additional concern is that aluminum is more expensive than steel.

Case (3): Look at the images above. The beam on the right above is made of steel.

Since we are comparing steel to steel, our requirement for the same elastic deflection is:

$$I_3 = I_1$$

I_1 is computed from the formula given above:

$$\begin{aligned} I_1 &= (4.0)(6.0)^3/12 \\ I_1 &= 72.0 \text{ cm}^4 \end{aligned}$$

I_3 is computed by using the above formula twice – once as if the beam were solid, and again to subtract out the empty space. The problem at this point is that the height of the web is unknown. We need to select the height so that I_3 comes out to be 72.0 cm^4 . This is a trial-and-error calculation. (Spreadsheets do this conveniently, e.g., Goal Seek in Excel, and many scientific hand calculators have a SOLVE feature. Additional commentary is given at the end of this presentation.)

Here, we state that the result of this root search is:

$$h_3 = 6.6 \text{ cm (6.61809... from the root search),}$$

and thus

$$\begin{aligned} A_3 &= (4.0)(6.6) - (3.0)(4.6) \\ &= 12.6 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} m_{L3} &= (7900)(12.6 \times 10^{-4}) \\ &= 10.0 \text{ kg/m,} \end{aligned}$$

where h_3 is measured from the top surface to the bottom surface, including both flange thicknesses and the web height.

We now have a beam with nearly the weight saving of aluminum, made out of less expensive steel. This demonstrates a general rule of thumb: for elastic deformation, it is often more advantageous to change geometry rather than change materials.

Case (4): Look at the images above. The beam on the right is made of aluminum, again to have the same elastic deflection as the Case (1) steel beam.

This computation is a repeat of Case (3), but with a new target value of:

$$\begin{aligned} I_4 &= (2.86)(72.0) \\ I_4 &= 205.7 \text{ cm}^4 \end{aligned}$$

The results are:

$$h_4 = 10.0 \text{ cm (from root search)}$$

$$\begin{aligned} A_4 &= (4.0)(10.0) - (3.0)(8.0) \\ &= 16.0 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} m_{L4} &= (2700)(16.0 \times 10^{-4}) \\ &= 4.3 \text{ kg/m.} \end{aligned}$$

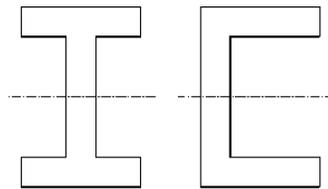
We have again approximately halved the weight of Cases (2) and (3), though in addition to more expensive aluminum, we now have our tallest beam. We will have to check that we have enough space in our application to accommodate it.

Summary

	Height (cm)	Mass per Length (kg/m)
Case (1) steel, rectangular	6.0	19.0
Case (3) steel, I-beam	6.6	10.0
Case (2) aluminum, rectangular	8.5	9.2
Case (4) aluminum, I-beam	10.0	4.3

Additional Commentary

Calculation of I for Cases (3) and (4)



The calculation of I about a horizontal axis is independent of the location of the web. The value of I for the beam on the right above is equal to that for the beam on the left. (Note: This is not true for the calculation of I about a vertical axis.)

Let thickness be designated as t for our special case in which all of the thicknesses are equal. The moment of area can be calculated as:

$$I = [bh^3/12] - [(b - t)(h - 2t)^3/12]$$

Since for all cases we are using $b = 4.0$ and $t = 1.0$, this reduces to:

$$I = h^3/3 - (h - 2)^3/4$$

The problem is to determine h such that $I = 72.0$ for Case (3) and $I = 205.7$ for Case (4).

Once h has been determined, the cross-sectional area is calculated in the same manner.

$$A = [bh] - [(b - t)(h - 2t)]$$

$$A = [h + 2(b - t)] t$$

With our values of $b = 4.0$ and $t = 1.0$, this reduces to:

$$A = (h + 6.0)(1.0)$$

