4.8 Odd Numbered Solutions

Section 4.1

1. Local maximums at $x = 3$, $x = 5$, $x = 9$, and $x = 13$. Global maximums at $x = 3$ and $x = 13$. Local minimums at $x = 1$, $x = 4.5$, $x = 7$, and $x = 10.5$. Global minimum at $x = 7$.

3. $f(x) = x^2 + 8x + 7$ so $f'(x) = 2x + 8$ which is defined for all values of $x$. $f'(x) = 0$ when $x = -4$ so $x = -4$ is a critical number. There are no endpoints. The only critical number is $x = -4$, and the only critical point is $(-4, f(-4)) = (-4, -9)$ which is the global (and local) minimum.

5. $f(x) = x^2 + 8x + 7$ so $f'(x) = 2x + 8$ which is defined for all values of $x$. $f'(x) = 0$ when $x = -4$ so $x = -4$ is a critical number. There are no endpoints. The only critical number is $x = -4$, and the only critical point is $(-4, f(-4)) = (-4, -9)$ which is the global (and local) minimum.

7. $f(x) = (x - 1)^2(x - 3)$ so $f'(x) = (x - 1)^2 + 2(x - 1)(x - 3) = (x - 1)(3x - 7)$ which is defined for all values of $x$. $f'(x) = 0$ when $x = 1$ and $x = 7/3$ so $x = 1$ and $x = 7/3$ are critical numbers. There are no endpoints. The only critical points are $(1, 0)$ which is a local maximum and $(7/3, -32/27)$ which is a local minimum. When the interval is the entire real number line, this function does not have a global maximum or global minimum.

9. $f(x) = 2x^3 - 96x + 42$ so $f'(x) = 6x^2 - 96$ which is defined for all values of $x$. $f'(x) = 0$ when $x = -4$ and $x = 4$ so $x = -4$ and $x = 4$ are critical numbers. There are no endpoints. The only critical points are $(-4, 298)$ which is a local maximum and $(4, -214)$ which is a local minimum. When the interval is the entire real number line, this function does not have a global maximum or global minimum.

11. $f(x) = 5x + \cos(2x + 1)$ so $f'(x) = 5 - 2\sin(2x + 1)$ which is defined for all values of $x$. $f'(x)$ is always positive (why?) so $f'(x)$ is never equal to 0. There are no endpoints. The function $f(x) = 5x + \cos(2x + 1)$ is always increasing and has no critical numbers, no critical points, no local or global maximums or minimums.

13. $f(x) = e^{-2(x-2)^2}$ so $f'(x) = -2(x-2)e^{-2(x-2)^2}$ which is defined for all values of $x$. $f'(x) = 0$ when $x = 2$ so $x = 2$ is a critical number. There are no endpoints. The only critical point is $(2, 1)$ which is a local and global maximum. When the interval is the entire real number line, this function does not have a local or global minimum.
15. See Fig. 3.1P15

17. \( f(x) = x^2 - 6x + 5 \) on \([-2, 5]\) so \( f'(x) = 2x - 6 \) which is defined for all values of \( x \). \( f'(x) = 0 \) when \( x = 3 \) so \( x = 3 \) is a critical number. The endpoints are \( x = -2 \) and \( x = 5 \) which are also critical numbers. The critical points are \((3, -4)\) which is the local and global minimum, \((-2, 21)\) which is a local and global maximum, and \((5, 0)\) which is a local maximum.

19. \( f(x) = 2 - x^3 \) on \([-1, 1]\) so \( f'(x) = -3x^2 \) which is defined for all values of \( x \). \( f'(x) = 0 \) when \( x = 0 \) so \( x = 0 \) is a critical number. The endpoints are \( x = -1 \) and \( x = 1 \) which are also critical numbers. The critical points are \((-2, 10)\) which is a local and global maximum, \((0,2)\) which is not a local or global maximum or minimum, and \((1,1)\) which is a local and global minimum.

21. \( f(x) = x^3 - 3x + 5 \) on \([-1, 1]\) so \( f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) \) which is defined for all values of \( x \). \( f'(x) = 0 \) when \( x = -1 \) and \( x = 1 \) so these are critical numbers. The endpoints \( x = -2 \) and \( x = 1 \) are also critical numbers. The critical points are \((-2, 3)\) which is a local and global minimum on \([-2, 1]\), the point \((-1, 7)\) which is a local and global maximum on \([-2, 1]\), and the point \((1, 3)\) which is a local and global minimum on \([-2, 1]\).

23. \( f(x) = x^5 - 5x^4 + 5x^3 + 7 \) so \( f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x - 3)(x - 1) \) which is defined for all values of \( x \). \( f'(x) = 0 \) when \( x = 0 \) and \( x = 1 \) in the interval \([0, 2]\) so each of these values is a critical number. The endpoints \( x = 0 \) and \( x = 2 \) are also critical numbers. The critical points are \((0, 7)\) which is a local minimum, \((1, 8)\) which is a local and global maximum, and \((2, -1)\) which is a local and global minimum. \(( f'(3) = 0 \) too, but \( x = 3 \) is not in the interval \([0, 2]\).)

25. \( f(x) = \frac{1}{x^2 + 1} \) so \( f'(x) = -\frac{2x}{(x^2 + 1)^2} \) which is defined for all values of \( x \). \( f'(x) = 0 \) when \( x = 0 \) but \( x = 0 \) is not in the interval \([1, 3]\) so \( x = 0 \) is not a critical number. The endpoints \( x = 1 \) and \( x = 3 \) are critical numbers. The critical points are \((1, 1/2)\) which is a local and global maximum, and \((3, 1/10)\) which is a local and global minimum.

27. \( A(x) = 4x \sqrt{1 - x^2} \) \((0 < x < 1)\).
\[
A'(x) = 4 \left[ \frac{-x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} \right] = 4 \left[ \frac{1 - 2x^2}{\sqrt{1 - x^2}} \right] = \begin{cases} > 0 & \text{if } 0 < x < 1/\sqrt{2} \\ < 0 & \text{if } 1/\sqrt{2} < x < 1 \end{cases}
\]

A maximum is attained when \( x = 1/\sqrt{2} \) : \( A(1/\sqrt{2}) = 4 \cdot \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{2}} = 4 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 2.\)
29. \( V = x(8 - 2x)^2 \) for \( 0 < x < 4 \).

\[
V' = x(2)(8 - 2x)(-2) + (8 - 2x)^2 = (8 - 2x)(-4x + 8 - 2x) = (8 - 2x)(8 - 6x) = 4(4 - x)(4 - 3x)
\]
so
\[
V' \begin{cases} < 0 & \text{if } 4/3 < x \\ > 0 & \text{if } 0 < x < 4/3. \end{cases}
\]

\[
V(4/3) = \frac{4}{3}(8 - \frac{8}{3})^2 = \frac{1024}{27} \approx 37.926 \text{ cubic units is the largest volume.}
\]

Smallest volume is 0 which occurs when \( x = 0 \) and \( x = 4 \).

31. (a) 4. The endpoints and two values of \( x \) for which \( f'(x) = 0 \).
(b) 2. The endpoints.
(c) At most \( n + 1 \). The 2 endpoints and the \( n - 1 \) interior points \( x \) for which \( f'(x) = 0 \).
At least 2. The 2 endpoints.

33. (a) local minimum at \( (1,5) \) (b) no extrema at \( (1,5) \)
(c) local maximum at \( (1,5) \) (d) no extrema at \( (1,5) \)

35. (a) \( 0, 2, 6, 8, 11, 12 \)
(b) \( 0, 6, 11 \)
(c) \( 2, 8, 12 \)

37. If \( f \) does not attain a maximum on \([a,b]\) or \( f \) does not attain a minimum on \([a,b]\), then \( f \) must have a discontinuity on \([a,b]\).

39. (a) yes, \(-1\) (b) no (c) yes, \(-1\) (d) no (e) yes, \(1 - \pi\)

41. (a) yes, 0 (b) yes, 0 (c) yes, 0 (d) yes, 0 (e)

43. (a) \( S(x) \) is minimum when \( x \approx 8 \).
(b) \( S(x) \) is maximum when \( x = 2 \).

Section 4.2

1. \( c \approx 3, 10, \text{ and } 13. \) (a) \( c = \pi/2 \) (b) \( c = 3\pi/2, 5\pi/2, 7\pi/2, 9\pi/2 \)

5. Rolle's Theorem asserts that the velocity \( h'(t) \) will equal 0 at some point between the time the ball is tossed and the time it comes back down. The ball is not moving as fast when it reaches the balcony from below.

7. The function does not violate Rolle's Thm. because the function does not satisfy the hypotheses of the theorem: \( f \) is not differentiable at 0, a point in the interval \(-1 < x < 1\).

9. No. The velocity is not the same as the rate of change of altitude, since altitude is only one of the components of position. Rolle's Theorem only says there was a time when my altitude was not changing.

11. Since \( f'(x) = 3x^2 + 5 \), \( f'(x) = 0 \) has no real roots. If \( f(x) = 0 \) for a value of \( x \) other than 2, then by the corollary from Problem 8, we would have an immediate contradiction.

13. (a) \( f(0) = 0, f(2) = 4, f'(c) = 2c \).
(b) \( f(1) = 4, f(5) = 8, f'(c) = 2c - 5 \).

14. (a) \( f(0) = 0, f(\pi/2) = 1, f'(c) = \cos(c) \).
(b) \( f(-1) = -1, f(3) = 27, f'(c) = 3c^2 \).
15. (a) $f(1) = 4$, $f(9) = 2$, $f'(c) = \frac{-1}{2\sqrt{c}} \cdot \frac{2-4}{1-9} = \frac{-1}{2\sqrt{c}} \cdot \frac{-1}{4} = \frac{-1}{2\sqrt{c}}$ implies $\frac{-1}{4} = \frac{-1}{2\sqrt{c}}$ so $c = 4$.
   (b) $f(1) = 3$, $f(7) = 15$, $f'(c) = 2$. $\frac{15-3}{7-1} = 2$ so any $c$ between 1 and 7 will do.

17. The hypotheses are not all satisfied since $f'(x)$ does not exist at $x = 0$ which is between $-1$ and 3.

19. Guilty. All we know is that $f'(c) = 17$ at some point, but this does not prove that the motorist "could not have been speeding."

21. $f(x) = x^3 + x^2 + 5x + c$. $f(1) = 7 + c = 10$ when $c = 3$. Therefore, $f(x) = x^3 + x^2 + 5x + 3$.

23. (a) $f'(x) = 2Ax$. We need $A(1)^2 + B = 9$ and $2A(1) = 4$ so $A = 2$ and $B = 7$ and $f(x) = 2x^2 + 7$.
   (b) $A(2)^2 + B = 3$ and $2A(2) = -2$ so $A = -1/2$ and $B = 5$ and $f(x) = \frac{-1}{2}x^2 + 5$.
   (c) $A(0)^2 + B = 2$ and $2A(0) = 3$. There is no such $A$. The point (0,2) is not on the parabola $y = x^2 + 3x - 2$.

25. $f(x) = x^3 + C$, a family of "parallel" curves for different values of $C$.

27. $v(t) = 300$. Assuming the rocket left the ground at $t = 0$, we have $y(1) = 300$ ft, $y(2) = 600$ ft, $y(5) = 1500$ ft.

29. $f''(x) = 6$, $f'(0) = 4$, $f(0) = -5$. $f(x) = 3x^2 + 4x - 5$.

31. (a) $A(x) = 3x$ 
   (b) $A'(x) = 3$.

33. (a) $A(x) = x^2 + x$
   (b) $A'(x) = 2x + 1$.

35. $a_1 = 5$, $a_2 = a_1 + 3 = 5 + 3 = 8$, $a_3 = a_2 + 3 = (5 + 3) + 3 = 11$, $a_4 = a_3 + 3 = (5 + 3) + 3 + 3 = 14$.
   In general, $a_n = 5 + 3(n - 1) = 2 + 3n$. 

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Section 4.3

1. See Fig. 3.3P1.

3. See Fig. 3.3P2.

4. See Fig. 3.3P3.

5. See Fig. 3.3P4.

6. See Fig. 3.3P5.


9. \( f'(x) = \frac{1}{x} > 0 \) for \( x > 0 \) so \( f(x) = \ln(x) \) is increasing on \( (0, \infty) \).

11. If \( f \) is increasing then \( f(1) < f(\pi) \) so \( f(1) \) and \( f(\pi) \) cannot both equal 2.

13. (a) \( x = 3, \ x = 8 \) (b) maximum at \( x = 8 \) (c) none (or only at right endpoint)

14. Relative maximum height at \( x = 2 \) and \( x = 7 \). Relative minimum height at \( x = 4 \).

15. Relative maximum height at \( x = 6 \). Relative minimum height at \( x = 8 \).

17. \( f(x) = x^3 - 3x^2 - 9x - 5 \) has a relative minimum at \( (3, -32) \) and a relative maximum at \( (-1, 0) \).

19. \( h(x) = x^4 - 8x^2 + 3 \) has a relative maximum at \( (0, 3) \) and relative minimums at \( (2, -13) \) and \( (-2, -13) \).

21. \( r(t) = 2(t^2 + 1)^{-1} \) has a relative maximum at \( (0, 2) \) and no relative minimums.

23. No positive roots. \( f(x) = 2x + \cos(x) \) is continuous. \( f(0) = 1 > 0 \). Since \( f'(x) = 2 - \sin(x) > 0 \) for all \( x \), \( f \) is increasing and never decreases back to the \( x \)-axis (a root).
24. One positive root. \( g(x) = 2x - \cos(x) \) is continuous. \( g(0) = -1 < 0 \) and \( g(1) = 2 - \cos(1) > 0 \) so by the Intermediate Value Theorem \( g \) has a root between 0 and 1. Since \( g'(x) = 2 + \sin(x) > 0 \) for all \( x \), \( g \) is increasing and can have only that 1 root.

25. \( h(x) = x^3 + 9x - 10 \) and \( h(1) = 0 \). \( h'(x) = 3x^2 + 9 = 3(x^2 + 3) > 0 \) for all \( x \) so \( h \) is always increasing and can cross the \( x \)-axis at most at one place. Since the graph of \( h \) crosses the \( x \)-axis at \( x = 1 \), that is the only root of \( h \).

27. \[
\begin{align*}
\text{(a)} & \quad f'(1) \text{ undefined} \\
\text{(b)} & \quad f(x) = x^3 + 1 \\
\text{(c)} & \quad \text{If they travel at the same positive speed in different directions, then their distance apart is not constant.}
\end{align*}
\]

Fig. 3.3P27

29. (a) \( h(x) = x^2, x^2 + 1, x^2 - 7 \), or, in general, \( x^2 + C \) for any constant \( C \).
(b) \( f(x) = x^2 + C \) for some value \( C \) and \( 20 = f(3) = 3^2 + C \) so \( C = 20 - 9 = 11 \). \( f(x) = x^2 + 11 \).
(c) \( g(x) = x^2 + C \) for some value \( C \) and \( 7 = g(2) = 2^2 + C \) so \( C = 7 - 4 = 3 \). \( g(x) = x^2 + 3 \).

Section 4.4

1. (a) \( f(t) = \) number of workers unemployed at time \( t \). \( f'(t) > 0 \) and \( f''(t) < 0 \)
(b) \( f(t) = \) profit at time \( t \). \( f'(t) < 0 \) and \( f''(t) > 0 \).
(c) \( f(t) = \) population at time \( t \). \( f'(t) > 0 \) and \( f''(t) > 0 \).

3. See Fig.

Fig. 3.3P3

5. (a) Concave up on \((0, 2), (2, 3+), (6, 9)\). Concave down on \((3+, 6)\). (A small technical note: we have defined concavity only at points where the function is differentiable, so we exclude the endpoints and points where the function is not differentiable from the intervals of concave up and concave down.)

7. \( g(x) = x^3 - 3x^2 - 9x + 7 \). \( g''(x) = 6x - 6 \). \( g''(-1) < 0 \) so \((-1, 12)\) is a local maximum. \( g''(3) > 0 \) so \((3, -20)\) is a local minimum.

9. \( f(x) = \sin^5(x) \). \( f''(x) = 5\{ -\sin^5(x) + 4\sin^3(x)\cos^2(x) \} \). \( f''(\pi/2) < 0 \) so \((\pi/2, 1)\) is a local maximum. \( f''(3\pi/2) > 0 \) so \((3\pi/2, -1)\) is a local minimum.

11. d and e.
Contemporary Calculus  
Dale Hoffman (2012)

15.  
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17.  
See Fig. 3.4P17.

Section 4.5

1.  
(a)  
2x + 2y = 200 so y = 100 − x. Maximize A = x·y = x(100 − x) = 100x − x^2.

\[ A' = 100 - 2x \text{ and } A' = 0 \text{ when } x = 50 \ (y = 100 - x = 50). \ A'' = -2 < 0 \]

x = 50 yields the maximum enclosed area. When x = 50,

\[ A = 50(100 - 50) = 2500 \text{ square feet}. \]

(b)  
2x + 2y = P so y = P/2 − x. Maximize A = x·y = x(P/2 − x) = (P/2)x − x^2.

\[ A' = P/2 - 2x \text{ and } A' = 0 \text{ when } x = P/4 \ (then \ y = P/2 - x = P/4). \ A'' = -2 < 0 \]

so x = P/4 yields the maximum enclosed area. This garden is a P/4 by P/4 square.

(c)  
2x + y = P so y = P − 2x. Maximize A = xy = x(P − 2x) = Px − 2x^2.

\[ A' = P - 4x \text{ and } A' = 0 \text{ when } x = P/4 \ (then \ y = P - 2x = P/2). \]

(d)  
A circle. A semicircle.

3.  
(a)  
120 = 2x + 5y so y = 24 − \(\frac{2}{5}\) x. Maximize A = xy = x(24 − \(\frac{2}{5}\) x) = 24x − \(\frac{2}{5}\) x^2.

\[ A' = 24 - \frac{4}{5} x \text{ and } A' = 0 \text{ when } x = 30 \ (then \ y = 12). \ A'' = -4/5 < 0 \]

so x = 30 yields the maximum enclosed area. Area is (30 ft)(12 ft) = 360 square feet.

(b)  
A circular pen divided into 4 equal stalls by two diameters shown in diagram (a) does a better job than a square.

\[ r = 120/(4 + 2\pi) \approx 11.67. \]

The resulting enclosed area is \( A = \pi r^2 \approx \pi(11.67)^2 \approx 427.8 \text{ sq. ft}. \)

The pen shown in diagram (b) does even better. If each semicircle has radius r, then the figure uses \( 4\sqrt{2} r + 4\pi r = 120 \) feet of fence so \( r = 120/(4\sqrt{2} + 4\pi) \approx 6.858 \).

The resulting enclosed area is \( A = (\text{square}) + (\text{four semicircles}) = (2r)^2 + 4(\frac{1}{2} \pi r^2) \approx 445.90 \text{ sq. ft}. \)
5. \[ 2x + 2y = 10 \] so \[ y = 5 - x. \]
Maximize \[ V = xy(10 - 2x) = x(5 - x)(10 - 2x) = 50x - 20x^2 + 2x^3. \]
\[ V' = 50 - 40x + 6x^2 = 2(3x - 5)(x - 5) \] and \[ V' = 0 \] when \[ x = 5 \] and \[ x = 5/3. \] When \[ x = 5, \] then \[ V = 0, \] clearly not a maximum, so \[ x = 5/3. \] The dimensions of the box with the largest volume are \[ 5/3, 10/3, \] and \[ 20/3. \]

7. (a) \[ V = \pi r^2 h = 100 \] so \[ h = \frac{100}{\pi r^2}. \]
Minimize \[ C = 2(\text{top area}) + 5(\text{bottom area}) + 3(\text{side area}) \]
\[ = 2(\pi r^2) + 5(\pi r^2) + 3(2\pi r h) = 7\pi r^2 + 6\pi \left( \frac{100}{\pi r^2} \right) = 7\pi r^2 + \frac{600}{r}. \]
\[ C' = 14\pi r - \frac{600}{r^2} \] and \[ C' = 0 \] when \[ r = \frac{3}{\sqrt{600/(14\pi)}} \approx 2.39 \] (then \[ h = \frac{100}{\pi r^2} \approx 5.57). \]
(b) Let \[ k = \text{top + bottom rate} = 2\hat{e} + \text{the bottom rate} > 2\hat{e} + 5\hat{e} = 7\hat{e}. \] Minimize \[ C = k\pi r^2 + \frac{600}{r}. \]
\[ C' = 2k\pi r - \frac{600}{r^2} \] and \[ C' = 0 \] when \[ r = \frac{3}{\sqrt{600/(2k\pi)}}. \] If \[ k = 9, \] then \[ r \approx 2.20. \] If \[ k = 10, \] then \[ r \approx 2.12. \] As the cost of the bottom material increases, the radius of the least expensive cylindrical can decreases: the least expensive can becomes narrower and taller.

9. Time = distance/rate. Run distance = \( x \) \((0 \leq x \leq 60 \) Why?) so run time = \( x/8. \)
Swim distance = \[ \sqrt{40^2 + (60-x)^2} \] so swim time = \[ \frac{1}{2} \sqrt{40^2 + (60-x)^2} \] and the total time is \[ T = \frac{x}{8} + \frac{1}{2} \sqrt{40^2 + (60-x)^2}. \]
\[ T' = \frac{1}{8} + \frac{1}{2} \left( \frac{40^2 + (60-x)^2}{2\sqrt{40^2 + (60-x)^2}} \right) \cdot 1/2 \cdot 2(60-x)(-1) = \frac{1}{8} - \frac{60-x}{2\sqrt{40^2 + (60-x)^2}}. \]
\[ T' = 0 \] when \[ x = 60 + \frac{40}{\sqrt{15}}. \] The value \[ x = 60 + \frac{40}{\sqrt{15}} > 60 \] so the least total time occurs when \[ x = 60 - \frac{40}{\sqrt{15}} \approx 49.7 \text{ meters}. \] In this situation, the lifeguard should run about \( 5/6 \) of the way along the beach before going into the water.

11. (a) Consider a similar problem with a new town \( D^* \) located at the "mirror image" of \( D \) across the river (Fig. 3.5P11a). If the water works is built at any location \( W \) along the river, then the distances are the same from \( W \) to \( D \) and to \( D^*: \) \[ \text{dist}(W,D) = \text{dist}(W,D^*). \]
Then \[ \text{dist}(C,W) + \text{dist}(W,D) = \text{dist}(C,W) + \text{dist}(W,D^*). \] The shortest distance from \( C \) to \( D^* \) is a straight line (Fig. 3.5P11b), and this straight line gives similar triangles with equal side ratios: \[ \frac{x}{3} = \frac{10-x}{5} \] so \[ x = 15/4 = 3.75 \text{ miles}. \] A consequence of this "mirror image" view of the problem is that "at the best location \( W \) the angle of incidence \( \alpha \) equals the angle of reflection \( \beta. \)"

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(b) Minimize \[ C = 3000 \text{dist}(C,W) + 7000 \text{dist}(W,D) = 3000 \sqrt{x^2 + 9} + 7000 \sqrt{(10 - x)^2 + 25}. \]

\[
C' = \frac{3000x}{\sqrt{x^2 + 9}} + \frac{-7000(10 - x)}{\sqrt{(10 - x)^2 + 25}} \quad \text{so} \quad C' = 0 \quad \text{when} \quad \frac{3x}{\sqrt{x^2 + 9}} = \frac{7(10 - x)}{\sqrt{(10 - x)^2 + 25}} \quad \text{and} \quad x \approx 7.82 \text{ miles}. 
\]

As it becomes relatively more expensive to build the pipe from a point \( W \) on the river to \( D \), the cheapest route tends to shorten the distance from \( W \) to \( D \).

13. (a) Let \( x \) be the length of one edge of the square end. Then \( V = x^2(108 - 4x) = 108x^2 - 4x^3 \).

\[ V' = 216x - 12x^2 = 6x(18 - x) \quad \text{so} \quad V' = 0 \quad \text{when} \quad x = 0 \text{ or } x = 18. \]

The dimensions of the greatest volume acceptable box with a square end are 18 by 18 by 36 inches: \( V = 11,664 \text{ in}^3 \).

(b) Let \( x \) be the length of the shorter edge of the end. Then \( V = 2x^2(108 - 6x) = 216x^2 - 12x^3 \).

\[ V' = 432x - 36x^2 = 36x(12 - x) \quad \text{so} \quad V' = 0 \quad \text{when} \quad x = 0 \text{ or } x = 12. \]

The dimensions of the largest box acceptable box with this shape are 12 by 24 by 36 inches: \( V = 10,368 \text{ in}^3 \).

(c) Let \( x \) be the radius of the circular end. Then \( V = \pi x^2(108 - 2\pi x) = 108\pi x^2 - 2\pi^2 x^3 \).

\[ V' = 216\pi x - 6\pi x^2 = 6\pi x(18 - \pi x) \quad \text{so} \quad V' = 0 \quad \text{when} \quad x = 0 \text{ or } x = 36/\pi \approx 11.46 \text{ inches}. \]

The dimensions of the largest box acceptable box with a circular end are a radius of \( 36/\pi \approx 11.46 \) and a length of 36 inches: \( V \approx 14,851 \text{ in}^3 \).

15. Without calculus: The area of the triangle is \( \frac{1}{2} \text{(base)(height)} = \frac{1}{2} (7) \text{(height)} \) and the height is maximum when the angle between the sides is a right angle.

Using calculus: Let \( \theta \) be the angle between the sides. Then the area of the triangle is

\[ A = \frac{1}{2} (7) \text{(height)} = \frac{1}{2} \left( \frac{10}{\sin \theta} \right) = \frac{10}{\sin \theta}. \]

\[ A' = -\frac{10 \cos \theta}{\sin^2 \theta} \quad \text{so} \quad A' = 0 \quad \text{when} \quad \theta = \pi/2, \quad \text{and the triangle is a right triangle with sides 7 and 10.} \]

Using either approach, the maximum area of the triangle is \( \frac{1}{2} (7)(10) = 35 \text{ square inches}, \) and the other side is the hypotenuse with length \( \sqrt{7^2 + 10^2} = \sqrt{149} \approx 12.2 \text{ inches.} \)

17. (a) \[ A = 2x(16 - x^2) = 32x - 2x^3. \]

Then \( A' = 32 - 6x^2 \quad \text{so} \quad A' = 0 \quad \text{when} \quad x = \sqrt{32}/6 \approx 2.31. \)

The dimensions are \( 2\sqrt[3]{32/6} \approx 4.62 \quad \text{and} \quad 16 - (\sqrt[3]{32/6})^2 = 64/6 \approx 10.67. \)

(b) \[ A = 2x(\sqrt{1 - x^2}) \quad \text{Then} \quad A' = 2\left( \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}} \right) \quad \text{so} \quad A' = 0 \quad \text{when} \quad x = 1/\sqrt{2} \approx 0.707. \]

The dimensions are \( 2(1/\sqrt{2}) \approx 1.414 \quad \text{and} \quad 1/\sqrt{2} \approx 0.707. \)

(c) The graph of \( |x| + |y| = 1 \) is a "diamond" (a square) with corners at \( (1,0), (0,1), (-1,0) \) and \( (0,-1) \).

For \( 0 \leq x \leq 1, A = 2x(1 - x) \quad \text{so} \quad A = 4x - 4x^2. \)

Then \( A' = 4 - 8x \) and \( A' = 0 \quad \text{when} \quad x = 1/2. \quad A'' = -8 \quad \text{so we have a local max.} \quad \text{The dimensions are} \quad 2(1/2) = 1 \quad \text{and} \quad 2(1 - 1/2) = 1. \)

(d) \[ A = 2x \cos(x) \quad (0 \leq x \leq \pi/2). \quad \text{Then} \quad A' = 2 \cos(x) - 2x \sin(x) \quad \text{so} \quad A' = 0 \quad \text{when} \quad x \approx 0.86. \quad \text{The dimensions are} \quad 2(0.86) = 1.72 \quad \text{and} \cos(0.86) \approx 0.65. \]
19. \( A = 6 \sin(\theta/2) \cdot 6 \cos(\theta/2) = 36 \cdot \frac{1}{2} \sin(\theta) = 18 \sin(\theta) \) and this is a maximum when \( \theta = \pi/2 \). Then the maximum area is \( A = 18 \sin(\pi/2) = 18 \) square inches. (This problem is similar to problem 15.)

21. \( V = \frac{1}{3} \pi r^2 h \) and \( h = \sqrt{9 - r^2} \) so \( V = \frac{1}{3} \pi r^2 \sqrt{9 - r^2} = \frac{\pi}{3} \sqrt{9r^4 - r^6} \). Then

\[
V' = \frac{\pi}{3} \left( 36r^3 - 6r^5 \right) \sqrt{9r^4 - r^6},
\]
and \( V' = 0 \) when \( 36r^3 = 6r^5 \) so \( r = \sqrt[3]{6} \approx 2.45 \) inches and

\[
h = \sqrt{9 - r^2} = 1.73 \text{ inches.}
\]

23. Let \( n \geq 10 \) be the number of passengers. The income is \( I = n(30 - (n - 10)) = 40n - n^2 \). The cost is \( C = 100 + 6n \) so the profit is \( P = I - C = (40n - n^2) - (100 + 6n) = 34n - n^2 - 100 \).

\( P' = 34 - 2n \) and \( P' = 0 \) when \( n = 17 \). 17 passengers on the flight maximize your profit.

(This is an example of treating a naturally discrete variable, the number of passengers, as a continuous variable.)

25. Apply the result of problem 24 with \( R = f \) and \( E = g \).

27. (i) Let \( D \) = diameter of the base of the can, and let \( H \) = the height of the can.

Then \( \theta = \arctan\left( \frac{\text{radius of can}}{\text{height of cg}} \right) = \arctan\left( \frac{D/2}{H/2} \right) \).

For this can, \( D = 5 \text{ cm} \) and \( H = 12 \text{ cm} \) (sorry this should be in the statement of the problem) so

\( \theta = \arctan(2.5/6) = \arctan(0.42) \approx 0.395 \) which is about \( 22.6^\circ \). The can can be tilted about \( 22.6^\circ \) before it falls over.

(ii) \( C(x) = \frac{360 + 9.6x^2}{60 + 19.2x} \) so \( C'(x) = \frac{(60 + 19.2x)(19.2x) - (360 + 9.6x^2)(19.2)}{(0 + 19.2x)^2} \). \( C'(x) = 0 \) when

\( (19.2)(9.6x^2 + 60x - 360) = 0 \) so \( x = 3.75 \): the height of the cola is \( h = 3.75 \text{ cm} \).

(iii) \( C(3.75) = 3.75 \) (The center of gravity is exactly at the top edge of the cola. It turns out that when the \( cg \) of a can and liquid system is as low as possible then the \( cg \) is at the top edge of the liquid.) Then \( \theta = \arctan(\text{radius/(height of cg)}) = \arctan(2.5/3.75) \approx 0.588 \) which is about \( 33.7^\circ \). In this situation, the can can be tilted about \( 33.7^\circ \) before it falls over.

(iv) Less.

28. (a) & (b) See Fig. 3.5P28. In the solutions to these "shortest path" problems, the roads all meet at \( 120^\circ \) angles.
Contemporary Calculus
Dale Hoffman (2012)

29. (a) $A = \text{(base)(height)} = (1 - x)(x^2) = x^2 - x^3$ for $0 \leq x \leq 1$. $A' = 2x - 3x^2 = 0$ if $x = 2/3$.
   (Clearly the endpoints $x=0$ and $x=1$ will not give the largest area.) Then $A = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{27}$.
   
   (b) $A = \text{(base)(height)} = (1 - x)(Cx^2) = Cx^2 - Cx^3$ for $0 \leq x \leq 1$.
   $A' = 2C - 3Cx^2 = 0$ if $x = 2/3$. Then $A = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4C}{27}$.
   
   (c) $A = \text{(base)(height)} = (B - x)(Cx^2) = BCx^2 - Cx^3$ for $0 \leq x \leq 1$.
   $A' = 2BC - 3Cx^2 = 0$ if $x = \frac{2}{3}B$. Then $A = \left(\frac{B}{3}\right)\left(\frac{2B}{3}\right)^2 = \frac{4B^2C}{27}$.

31. (a) $y = 20 - \frac{20}{50} x$. $A = \text{(base)(height)} = xy = x(20 - \frac{2}{5}x) = 20x - \frac{2}{5}x^2$.
   $A' = 20 - \frac{4}{5}x = 0$ when $x = 25$. Then $y = 10$ and Area = 250.
   
   (b) $y = H - \frac{H}{B} x$. $A = \text{(base)(height)} = x(H - \frac{H}{B} x) = Hx - \frac{H}{B}x^2$.
   $A' = H - \frac{2H}{B}x = 0$ when $x = \frac{B}{2}$. Then $y = \frac{H}{2}$ and Area = $\frac{BH}{4}$.

33. $F = \text{cost} = \text{(top cost)} + \text{(bottom)} + \text{(sides)} = (\pi r^2)A + (\pi r^2)B + (2\pi rh)C$
   But we know $V = \pi r^2 h$ so $h = \frac{V}{\pi r^2}$ so $F = \pi r^2 (A + B) + \frac{2CV}{r}$.
   Then $F' = 2\pi r(A + B) - \frac{2CV}{r^2} = 0$ when $r = \sqrt{\frac{CV}{\pi(A + B)}}$. Now you can find $h$ and $F$.

Section 4.6

1. (a) $h$ has a root at $x = 1$.
   
   (b) limits of $h(x) = f(x)/g(x)$: as $x \to 1^+$ is 0: as $x \to 1^-$ is 0: as $x \to 3^+$ is $-\infty$: as $x \to 3^-$ is $+\infty$
   
   (c) $h$ has a vertical asymptote at $x = 3$.

3. limits of $h(x) = f(x)/g(x)$: as $x \to 2^+$ is $+\infty$: as $x \to 2^-$ is $-\infty$: as $x \to 4^+$ is 0: as $x \to 4^-$ is 0

5. 0 7. –3 9. 0 11. DNE
13. 2/3 15. 0 17. –7 19. 0
21. cos(0) = 1 23. ln(1) = 0
25. (a) $V(t) = 50 + 4t$ gallons, and $A(t) = 0.8t$ pounds of salt
(b) \[ C(t) = \frac{\text{amount of salt}}{\text{total amount of liquid}} = \frac{A(t)}{V(t)} = \frac{0.8t}{50 + 4t} \]

(c) "after a long time" (as \( t \to \infty \)), \( C(t) \to 0.8/4 = 0.2 \) pounds of salt per gallon.

(d) \( V(t) = 200 + 4t \), \( A(t) = 0.8t \), \( C(t) = \frac{0.8t}{200 + 4t} \to 0.8/4 = 0.2 \) pounds of salt per gallon.

27. \(+\infty\) 29. \(-\infty\) 31. \(-\infty\) 33. \(-\infty\) 35. \(+\infty\) 37. \(-\infty\) 39. 1 41. \(-\infty\)

42. Horizontal: \( y = 1 \). Vertical: \( x = 1 \).

44. Horizontal: \( y = 0 \). Vertical: \( x = 0 \) (a "hole" at \( x = 1 \)).

46. Horizontal: \( y = 1/3 \). Vertical: \( x = 1 \).

48. Horizontal: \( y = 0 \). Vertical: \( x = 0 \).

50. Horizontal: \( y = 0 \). Vertical: \( x = 1 \). (and a "hole" at \( x = 0 \))

51. \( y = 2x + 1 \). \( x = 0 \)

52. \( y = x \)

53. \( y = \sin(x) \). \( x = 2 \)

54. \( y = x \)

55. \( y = x^2 \)

56. \( y = x^2 + 1 \). \( x = -1 \)

57. \( y = \cos(x) \). \( x = 3 \)

58. \( y = x \)

59. \( y = \sqrt{x} \). \( x = -3 \).

Section 4.7

1. \( 3/2 \)

3. \( 3/5 \)

5. \(-1 \)

7. 0

9. 0

11. 9/2

13. For \( a \neq 0 \):
\[
\frac{f'}{g'} = \frac{m}{n} \frac{x^{(m-n)}}{x^{(m-n)}} \to \frac{m}{n} a^{(m-n)}
\]

For \( a = 0 \),
\[
\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \begin{cases} 
0 & \text{if } m > n \\
1 & \text{if } m = n \\
+\infty & \text{if } m < n \text{ and } (m-n) \text{ is even} \\
\text{DNE} & \text{if } m < n \text{ and } (m-n) \text{ is odd} 
\end{cases}
\]

15. 0

17. \( \frac{f'}{g'} = \frac{pe^{px}}{3} \to \frac{p}{3} \) so \( p = 3(5) = 15 \).

19. (a) All three limits are \(+\infty\).

(b) After applying L'Hopital's Rule \( d \) times,
\[
\frac{f^{(d)}}{g^{(d)}} = \frac{a^b + b^a}{c(d-1)(d-2) \cdots (2)(1)} = \text{constant} \cdot e^{bn}
\]

\( \to +\infty \).
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