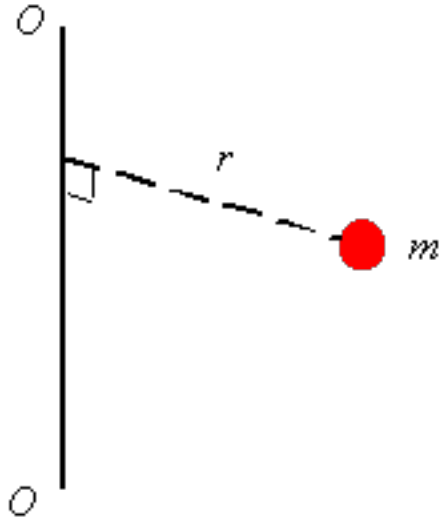


Mass moment of inertia

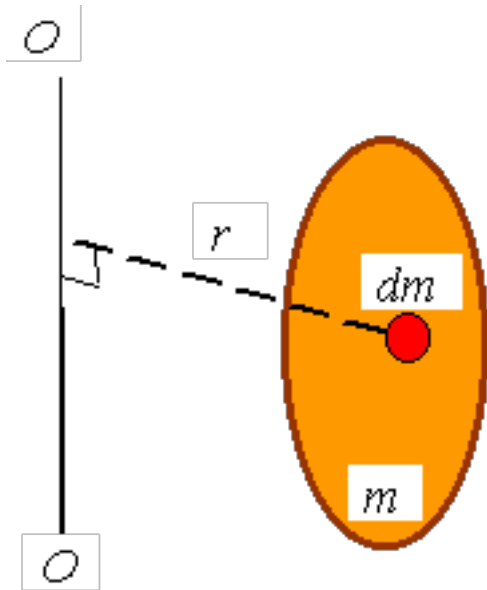
Mass moment of inertia for a particle: The mass moment of inertia is one measure of the distribution of the mass of an object relative to a given axis. The mass moment of inertia is denoted by I and is given for a single particle of mass m as

$$I_O = r^2 m$$



where $O-O$ is the axis around which one is evaluating the mass moment of inertia, and r is the perpendicular distance between the mass and the axis $O-O$. As can be seen from the above equation, the **mass moment of inertia has the units of mass times length squared**. The mass moment of inertia should not be confused with the area moment of inertia which has units of length to the power four. Mass moments of inertia naturally appear in the equations of motion, and provide information on how difficult (how much inertia there is) it is to rotate the particle around a given axis.

Mass moment of inertia for a rigid body: When calculating the mass moment of inertia for a rigid body, one thinks of the body as a sum of particles, each having a mass of dm . Integration is used to sum the moment of inertia of each dm to get the mass moment of inertia of the body. The equation for the mass moment of inertia of the rigid body is



$$I = \sum r^2 dm = \int r^2 dm$$

The integration over mass can be replaced by integration over volume, area, or length. For a fully three dimensional body using the density ρ one can relate the element of mass to the element of volume. In this case the density has units of mass per length cubed and the relation is given as

$$dm = \rho dV$$

and the equation for the mass moment of inertia becomes

$$I = \int_V r^2 \rho dV$$

The integral is actually a triple integral. If the coordinate system used is rectangular then $dV = dx dy dz$. If the coordinates uses are cylindrical coordinates then $dV = r dr d\theta dz$.

For a two dimensional body like a plate or a shell one can use density ρ per unit area (units of mass per length squared) to change the integration using the relation

$$dm = \rho dA$$

where A is the surface area and dA differential element of area. For example, for rectangular coordinates $dA = dx dy$ and for polar coordinates $dA = r dr d\theta$. After this substitution one gets the equation to calculate the mass moment of inertia as

$$I = \int_A r^2 \rho dA$$

If the body is a rod like object then one can use the relation

$$dm = \rho dl$$

to get

$$I = \int_l r^2 \rho dl$$

where l is a coordinate along the length of the rod and the density ρ is in units of mass per unit length.

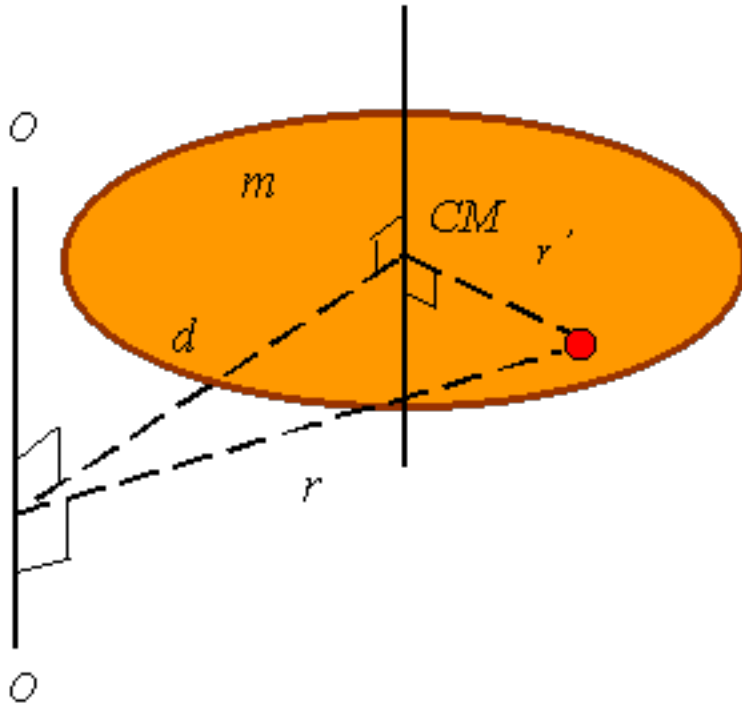
Radius of gyration: Sometime in place of the mass moment of inertia the radius of gyration k is provided. The mass moment of inertia can be calculated from k using the relation

$$I = mk^2$$

where m is the total mass of the body. One can interpret the radius of gyration as the distance from the axis that one could put a single particle of mass m equal to the mass of the rigid body and have this particle have the same mass moment of inertia as the original body.

Parallel-axis theorem: The moment of inertia around any axis can be calculated from the moment of inertia around parallel axis which passes through the center of mass. The equation to calculate this is called the parallel axis theorem and is given as

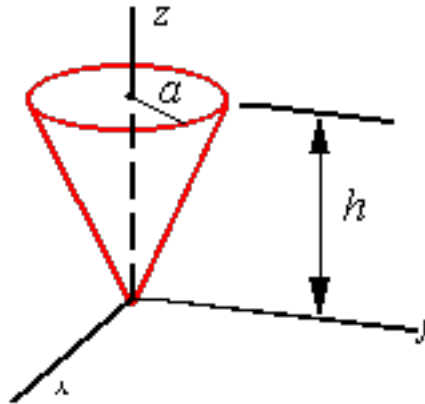
$$I = \bar{I} + md^2$$



where d is the distance between the original axis and the axis passing through the center of mass, m is the total mass of the body, and \bar{I} is the moment of inertia around the axis passing through the center of mass.

Composite bodies: If a body is composed of several bodies, to calculate the moment of inertia about a given axis one can simply calculate the moment of inertia of each part around the given axis and then add them to get the mass moment of inertia of the total body.

EXAMPLE 1: MASS MOMENT OF INERTIA



Calculate the mass moment of inertia of the cone about the z -axis. Assume the cone is made of a uniform material of density ρ (mass per unit volume).

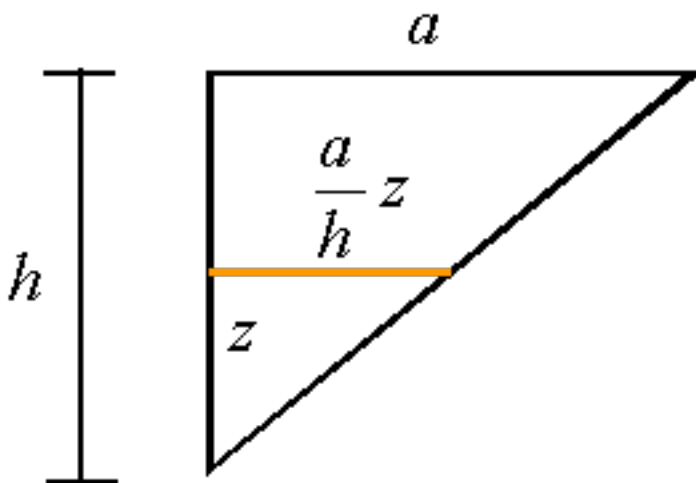
Solution:

The mass moment of inertia about the z -axis is given by

$$I_{zz} = \int_{\mathcal{V}} r^2 dm = \int_{\mathcal{V}} r^2 \rho dV$$

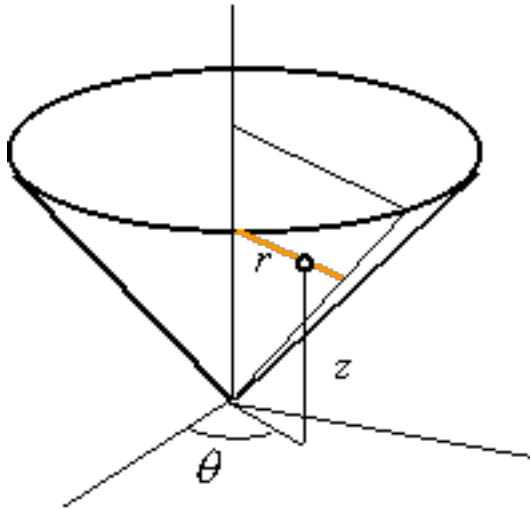
The element of volume in a cylindrical coordinate system is given by

$$dV = r dr d\theta dz$$



The domain of the cone in cylindrical coordinates is defined by

Source URL: <http://emweb.unl.edu/negahban/em223/note19/ex3/ex3.htm>
 Saylor URL: <http://www.saylor.org/courses/ME102>



$$\begin{cases} 0 \leq z \leq h \\ 0 \leq r \leq \frac{a}{h}z \\ 0 \leq \theta \leq 2\pi \end{cases}$$

axis can be written as

Therefore, the mass moment of inertia about the z-

$$\begin{aligned} I_{zz} &= \int_V r^2 \rho dV = \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\frac{a}{h}z} r^3 \rho dr d\theta dz \\ &= \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} \frac{a^4}{4h^4} z^4 \rho d\theta dz \\ &= \int_{z=0}^{z=h} \frac{\pi a^4}{2h^4} z^4 \rho dz = \frac{\pi a^4 h \rho}{10} \end{aligned}$$

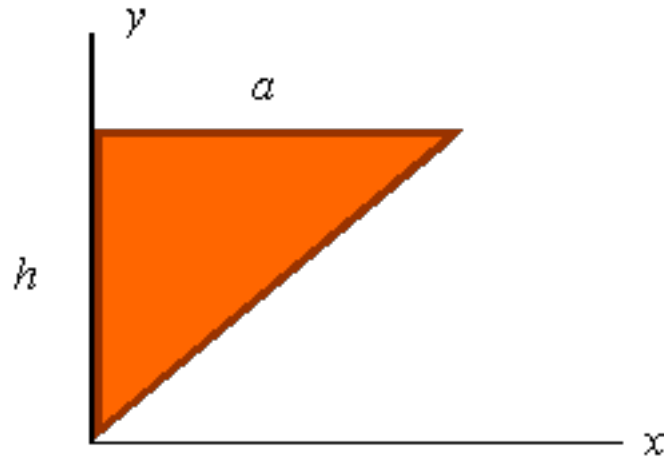
For a uniform cone the density can be calculated using the total mass and total volume of the cone so that

$$\rho = \frac{m}{V} = \frac{m}{\frac{1}{3}\pi a^2 h}$$

Therefore, the moment of inertia in terms of the total mass of the cone can be written as

$$I_{zz} = \frac{\pi a^4 h \rho}{10} = \frac{3 a^2 m}{10}$$

EXAMPLE 2: MASS MOMENT OF INERTIA



Calculate the mass moment of inertia of the triangular plate about the y -axis. Assume the plate is made of a uniform material and has a mass of m .

Solution:

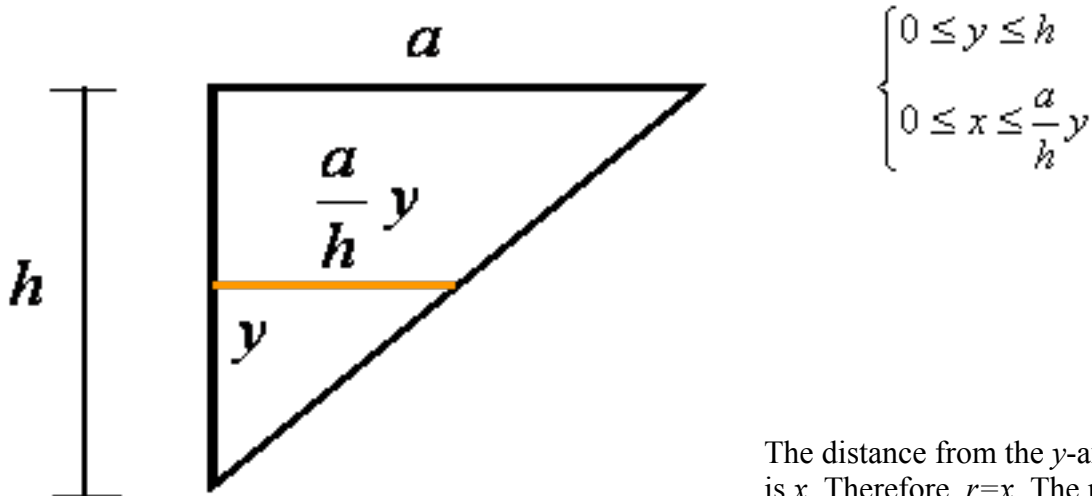
The mass moment of inertia about the y -axis is given by

$$I_{yy} = \int_{\mathcal{V}} r^2 dm = \int_A r^2 \rho dA$$

The element of area in rectangular coordinate system is given by

$$dA = dx dy = dy dx$$

The domain of the triangle is defined by



can be written as

The distance from the y -axis is x . Therefore, $r=x$. The mass moment of inertia about the y -axis

$$\begin{aligned}
 I_{yy} &= \int_A r^2 \rho \, dA = \int_{y=0}^{y=h} \int_{x=0}^{x=\frac{a}{h}y} x^2 \rho \, dx \, dy \\
 &= \int_{y=0}^{y=h} \frac{a^3}{3h^3} y^3 \rho \, dy \\
 &= \frac{a^3 h \rho}{12}
 \end{aligned}$$

For a uniform plate the density can be calculated using the total mass and total area of the plate so that

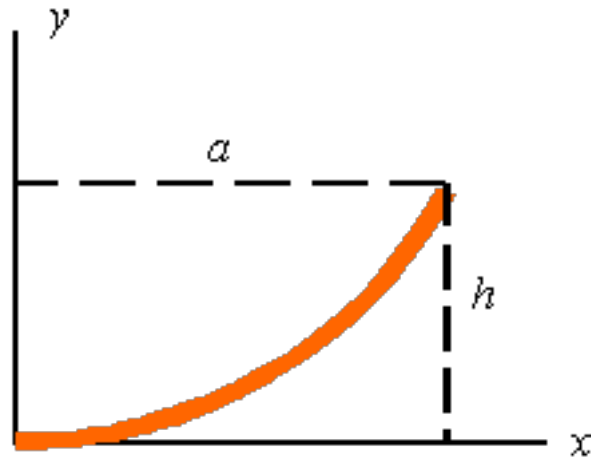
$$\rho = \frac{m}{A} = \frac{m}{\frac{1}{2}ah}$$

Therefore, the moment of inertia in terms of the total mass of the cone can be written as

$$I_{yy} = \frac{a^3 h \rho}{12} = \frac{a^2 m}{6}$$

EXAMPLE 3: MASS MOMENT OF INERTIA

Source URL: <http://emweb.unl.edu/negahban/em223/note19/ex3/ex3.htm>
 Saylor URL: <http://www.saylor.org/courses/ME102>



Calculate the mass moment of inertia of the parabolic rod about the y -axis. Assume the rod is made of a uniform material and has a mass of m .

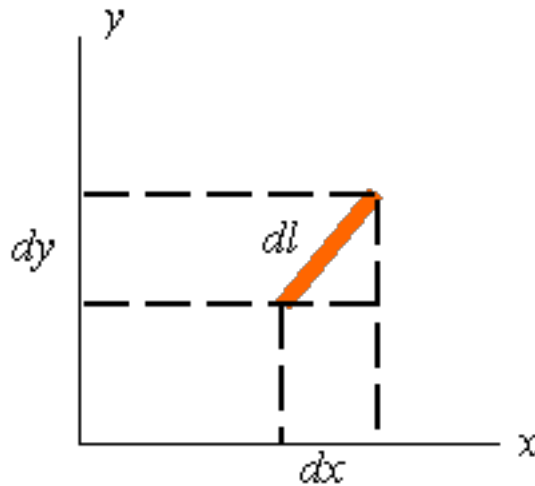
Solution:

The mass moment of inertia about the y -axis is given by

$$I_{yy} = \int_{\bar{x}} r^2 dm = \int_l r^2 \rho dl$$

The length of the bar can be calculated from

$$l = \int_l dl$$



The element of arc length in a rectangular coordinate system can be written as

$$dl = \sqrt{dx^2 + dy^2} = \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right] dx = \left[\sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \right] dy$$

The equation for the parabola is

$$y = kx^2$$

Substitution of the point (a, h) into this equation gives the equation of the bar as

$$y = \frac{h}{a^2} x^2$$

The length of the bar can, therefore, be calculated as

$$\begin{aligned}
 l &= \int_l dl = \int_{x=0}^{x=a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= \int_{x=0}^{x=a} \sqrt{1 + \left(\frac{2h}{a^2} x\right)^2} dx
 \end{aligned}$$

The distance from the y-axis is x . Therefore, $r=x$. The mass moment of inertia about the y-axis can be written as

$$\begin{aligned}
 I_{zz} &= \int_l r^2 \rho dl = \int_l x^2 \rho dl \\
 &= \rho \int_{x=0}^{x=a} x^2 \sqrt{1 + \left(\frac{2h}{a^2} x\right)^2} dx
 \end{aligned}$$

For a uniform bar the density can be calculated using the total mass and total length of the bar so that

$$\rho = \frac{m}{l}$$