Vectors: Review and Self Study

The exercises in this subunit are intended to refresh your memory and provide additional practice of skills that you will need in problem solving. For in depth coverage of the fundamentals, you may wish to refer to other resources in subunit 1.1 of this course and to review resources in the Saylor Foundation's Multivariable Calculus.

This review presents several exercises that you should complete before attempting additional study in ME102: Mechanics I. It is suggested that you read the study topics below and review elsewhere any areas in which you feel deficient. You should be especially aware of the representation of forces and torques as vector quantities.

What is a vector?

In some ways it is easiest to think of a vector as just an ordered set of numbers or other objects. In the context of mechanics, we most often associate those numbers with directions in a Euclidian space. These directions might be the orthonormal components of a right-handed, rectangular coordinate system (the **i**, **j**, **k** with which you are familiar), or some other directions. For the purposes of this review, we will work primarily with the rectangular coordinate system in which **i**, **j**, and **k** are the orthogonal unit vectors.

1**. Addition of vectors.** Given the two vectors **m** = a**i** + b**j** + c**k** and **p** = d**i** + f**j** + g**k**, the sum is given by **s** = (a + d)**i** + (b + f)**j** + (c + g)**k**. Addition of vectors is commutative. It is perhaps instructive to visualize vector summation in two dimensions, as in the picture below.

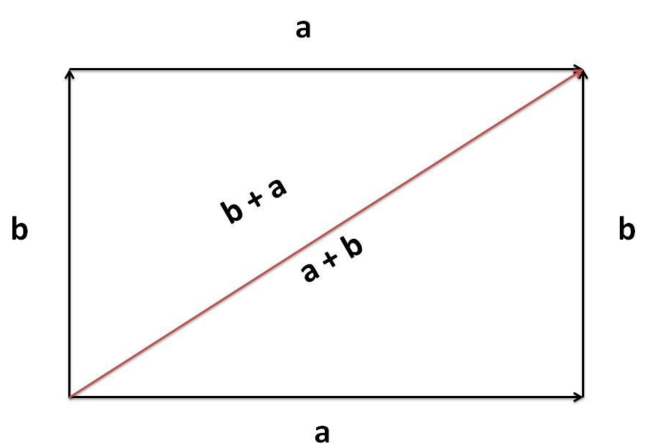


Illustration of addition of vectors a and b, and the cummutativity of the operation.

Example: The vectors m and n are given by **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k**. What is the sum of these two vectors?

Answer: s = 13**i** + 11**j** + 5**k**.

2. **Scalar multiplication.** Given a scalar h and a vector **m** = a**i** + b**j** + c**k**, the quantity h**m** = ha**i** + hb**j** + hc**k**. The operation of scalar multiplication is commutative.

3. **Subtraction of vectors.** Given the two vectors **m** = a**i** + b**j** + c**k** and **p** = d**i** + f**j** + g**k**, the difference **m** – **p** is given by scalar multiplication of n by –1 and addition.
s = (a – d)**i** + (b – f)**j** + (c – g)**k**. Subtraction of vectors is anticommutative (just like scalars). It is perhaps instructive to visualize vector subtraction in two dimensions, as in the picture below.

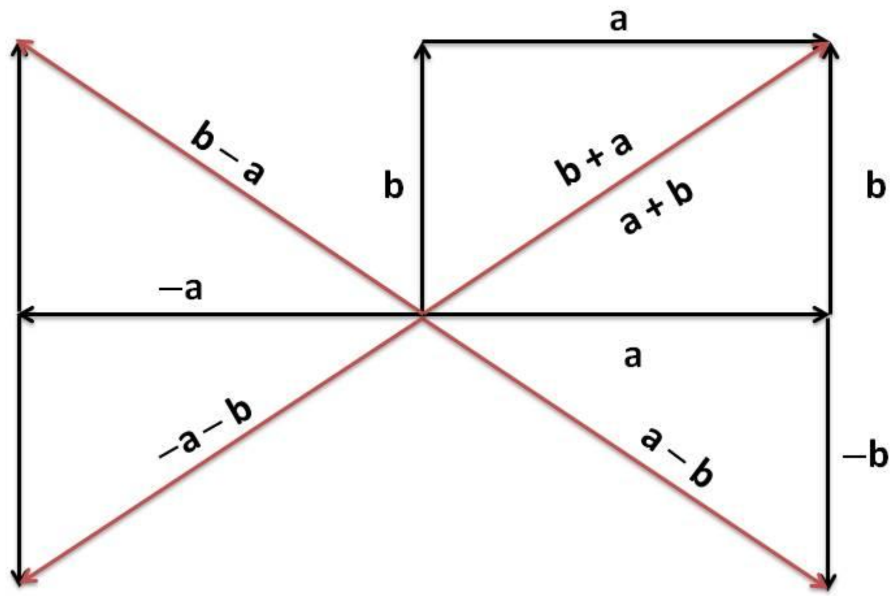


Illustration of vector negation (multiplication by –1) and vector subtraction in two dimensions.

Example: The vectors m and n are given by **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k.** What is the result of **s** = **p** – **m** of these two vectors?

Answer: **s** = – 3**i** – 3**j** – 1**k**.

4. **The dot or Gibbs scalar product of two vectors.** The dot product can be calculated in two ways. For the vectors **m** = a**i** + b**j** + c**k** and **p** = d**i** + f**j** + g**k,** the dot product s = |**m**| · |**p**| is given by ad + bf + cg. Note that |**m**| · |**p**| = |**p**| · |**m**|.

Examle: The vectors m and n are given by **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k.** What is the result of **m** · **p** for these two vectors?

Answer: s = 40 + 28 + 6 = 74.

5. **The magnitude of a vector.** The magnitude of a vector is a measure of its length, regardless of direction. The magnitude is a scalar obtained by the square root of the dot product of the vector with itself. For **m** = a**i** + b**j** + c**k**, we denote the magnitude with |**m**| = sqrt(a^2 + b^2 + c^2).

Example: For **m** = 5**i** + 4**j** + 2**k,** what is |**m**|?

Answer: sqrt(25 + 16 + 4) = 3 sqrt(5) = $3\sqrt{5}$.

6. **Angle between vectors and the dot product.** We may determine the angle between vectors (or vice versa) by making use of the relation, **m** · **p** = |**m**| · |**p**| cos θ.

Example: For the vectors **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k,** what is the angle between them?

Answer: First calculate **m** · **p** = 74 as was done above in number 4. Then calculate |**m**| = 3 sqrt(5) and |**p**| = sqrt(122). Thus cos θ = **m** · **p** / (|**m**| |**p**|)= 74/74.094, and θ = 0.05 radians.

7. **Cross product (or Gibbs vector product) of two vectors.** The cross product (or Gibbs vector product) of the two vectors **m** = a**i** + b**j** + c**k** and **p** = d**i** + f**j** + g**k** is given by the vector **s** = **m** x **p** = (bg – cf)**i** + (dc – ga)**j** + (af – bd)**k.** You may have seen one or more schemes for remembering the way in which the vector components combine to form a cross product. Some of these are listed in the Wikipedia article Cross Product. Development of a visual interpretation of a cross product is especially useful in interpreting problems involving torque and angular momentum.

Note that **m** x **p** = – **p** x **m**

Example: For the vectors **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k,** what is the cross product?

Answer: **m** x **p** = –2**i** + **j** + 3**k**.

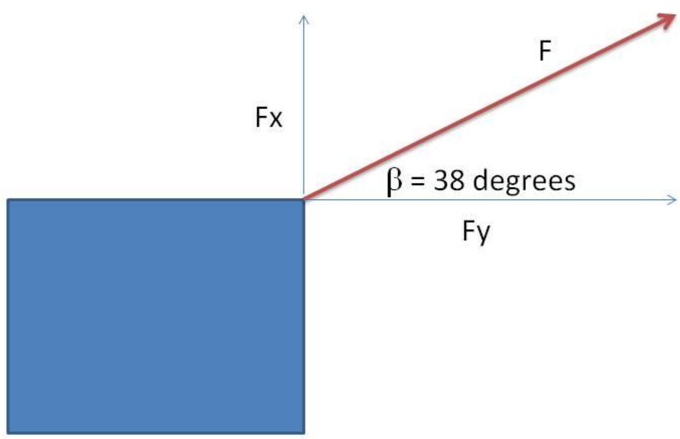
8. **Angle between vectors and the cross product**. The cross product can also be expressed as **m** x **p** = |**m**| |**p**| sin θ **n,** in which **n** is a unit vector orthogonal to the plane formed by vectors **m** and **p**. See here for a description of the right-handed convention. The direction of **n** is determined by convention.

Example: Solve for the angle between **m** = 5**i** + 4**j** + 2**k** and **p** = 8**i** + 7**j** + 3**k** using the cross product.

Answer: We can solve for the angle between vectors **m** and **p** from q = asin(|**m** x **p**| / (|**m**| |**p**|)) = asin(sqrt(14)/74.08) = 0.05 radians. Note that this calculation agrees with that in number 6 above.

9. **Conversion between rectangular and polar coordinates**. Often you may find it necessary to convert between representation of vectors in polar coordinates and vectors in rectangular coordinates.

Example: A person is tugging a sled behind him by pulling on a rope at a 38-degree angle with a force F, as shown in the figure. What are the horizontal and vertical components of that force?



Example: What are Fx and Fy as a fraction of F?

Answer: We have a force with magnitude F and direction 38 degrees expressed in polar coordinates. To convert to rectangular coordinates we use the trigonometric relations F_x = Fcos(38 degrees) and F_y = Fsin(38 degrees).