Deriving MRS from Utility Function, Budget Constraints, and Interior Solution of Optimization

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Outline
1. Chap 3: Utility Function, Deriving MRS
2. Chap 3: Budget Constraint
3. Chap 3: Optimization: Interior Solution

1 Utility Function, Deriving MRS

Examples of utility:

Example (Perfect substitutes).
\[ U(x, y) = ax + by. \]

Example (Perfect complements).
\[ U(x, y) = \min\{ax, by\}. \]

Example (Cobb-Douglas Function).
\[ U(x, y) = Ax^by^c. \]

Example (One good is bad).
\[ U(x, y) = -ax + by. \]

An important thing is to derive MRS.
\[ MRS = -\frac{dy}{dx} = \text{Slope of Indifference Curve}. \]
Figure 1: Utility Function of Perfect Substitutes

Figure 2: Utility Function of Perfect Complements

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Figure 3: Cobb-Douglas Utility Function

Figure 4: Utility Function of the Situation That One Good Is Bad
Because utility is constant along the indifference curve,

\[
    u = (x, y(x)) = C, \quad \implies \\
    \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0, \quad \implies \\
    -\frac{dy}{dx} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u}.
\]

Thus,

\[
    MRS = \frac{\partial x}{\partial y}.
\]

Example (Sample utility function).

\[
    u(x, y) = xy^2.
\]

Two ways to derive MRS:

- Along the indifference curve

\[
    xy^2 = C. \\
    y = \sqrt{\frac{C}{x}}.
\]

Thus,

\[
    MRS_d = -\frac{dy}{dx} = \frac{\sqrt{C}}{2x^{3/2}} = \frac{y}{2x}.
\]

- Using the conclusion above

\[
    MRS = \frac{\partial u}{\partial x} = \frac{y^2}{2xy} = \frac{y}{2x}.
\]

2 Budget Constraint

The problem is about how much goods a person can buy with limited income. Assume: no saving, with income \( I \), only spend money on goods \( x \) and \( y \) with the price \( P_x \) and \( P_y \).

Thus the budget constraint is

\[
    P_x \cdot x + P_y \cdot y \leq I.
\]

Suppose \( P_x = 2, P_y = 1, I = 8 \), then

\[
    2x + y \leq 8.
\]

The slope of budget line is

\[
    -\frac{dy}{dx} = \frac{P_x}{P_y}.
\]

Bundles below the line are affordable.

Budget line can shift:

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Figure 5: Budget Constraint

Figure 6: Budget Line Shifts Because of Change in Income
Figure 7: Budget Line Rotates Because of Change in Price

- Change in Income Assume $I' = 6$, then $2x + y = 6$. The budget line shifts right which means more income makes the affordable region larger.

- Change in Price Assume $P_x' = 2$, then $2x + 2y = 8$. The budget line changes which means lower price makes the affordable region larger.

3 Optimization: Interior Solution

Now the consumer’s problem is: how to be as happy as possible with limited income. We can simplify the problem into language of mathematics:

$$\max_{x,y} U(x, y) \text{ subject to } \begin{cases} \ xP_x + yP_y \leq I \\ x \geq 0 \\ y \geq 0 \end{cases}.$$ 

Since the preference has non-satiation property, only $(x, y)$ on the budget line can be the solution. Therefore, we can simplify the inequality to an equality:

$$xP_x + yP_y = I.$$ 

First, consider the case where the solution is interior, that is, $x > 0$ and $y > 0$. Example solutions:

- Method 1
Figure 8: Interior Solution to Consumer’s Problem

From Figure 8, the utility function reaches its maximum when the indifferent curve and constraint line are tangent, namely:

\[
\frac{P_x}{P_y} = MRS = \frac{\partial u}{\partial x} \frac{\partial x}{\partial y} = \frac{u_x}{u_y}.
\]

- If

\[
\frac{P_x}{P_y} > \frac{u_x}{u_y},
\]

then one should consume more \( y \), less \( x \).

- If

\[
\frac{P_x}{P_y} < \frac{u_x}{u_y},
\]

then one should consume more \( x \), less \( y \). Intuition behind \( \frac{P_x}{P_y} = MRS \):

\( \frac{P_x}{P_y} \) is the market price of \( x \) in terms of \( y \), and MRS is the price of \( x \) in terms of \( y \) valued by the individual. If \( \frac{P_x}{P_y} > MRS \), \( x \) is relatively expensive for the individual, and hence he should consume more \( y \). On the other hand, if \( \frac{P_x}{P_y} < MRS \), \( x \) is relatively cheap for the individual, and hence he should consume more \( x \).

- Method 2: Use Lagrange Multipliers

\[
L(x, y, \lambda) = u(x, y) - \lambda(xP_x + yP_y - I).
\]
In order to maximize $u$, the following first order conditions must be satisfied:

$$\frac{\partial L}{\partial x} = 0 \implies \frac{u_x}{P_x} = \lambda,$$
$$\frac{\partial L}{\partial y} = 0 \implies \frac{u_y}{P_y} = \lambda,$$
$$\frac{\partial L}{\partial \lambda} = 0 \implies xP_x + yP_y - I = 0.$$

Thus we have

$$\frac{P_x}{P_y} = \frac{u_x}{u_y}.$$

- Method 3

Since $xP_x + yP_y + I = 0$,

$$y = \frac{I - xP_x}{P_y}.$$

Then the problem can be written as

$$\max_{x,y} u(x, y) = u(x, I - \frac{xP_x}{P_y}).$$

At the maximum, the following first order condition must be satisfied:

$$u_x + u_y\left(\frac{\partial y}{\partial x}\right) = u_x + u_y\left(\frac{P_x}{P_y}\right) = 0.$$

$$\implies \frac{P_x}{P_y} = \frac{u_x}{u_y}.$$