Fraction versus rational number. What’s the difference? It’s not an easy question. In fact, the difference is somewhat like the difference between a set of words on one hand and a sentence on the other. A symbol is a fraction if it is written a certain way, but a symbol that represents a rational number is a rational number no matter how it is written. Here are some examples. The symbol $\frac{1}{\pi}$ is a fraction that is not a rational number. On the other hand $\frac{2}{3}$ is both a fraction and a rational number. Now 0.75 is a rational number that is not a fraction, so we have examples of each that is not the other. To get a little deeper, a fraction is a string of symbols that includes a fraction bar. Any real number can be written as a fraction (just divide by 1). But whether a number if rational depends on its value, not on the way it is written. What we’re saying is that in the case of fractions, we are dealing with a syntactic issue, and in case of rational numbers, a semantic issue, to borrow two terms from computer science. For completeness, we say that a number is rational if it CAN be represented as a quotient of two integers. So 0.75 is rational because we can find a pair of integers, 3 and 4, whose quotient is 0.75.

Here’s another way to think about the difference. Consider the question ‘Are these numbers getting bigger or smaller?’

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \]

This apparently amusing question can provoke some serious questions about what we mean by the word ‘number’. Indeed, there are two aspects of numbers that often get blurred together: the value of a number and the numeral we write for the number. By ‘number’ we usually mean the value while the word ‘numeral’ refers to the symbol we use to communicate the number. So the numbers above are getting smaller while the numerals are getting bigger. This contrast between symbol and substance also explains the difference between rational number and fraction. A fraction is a numeral while a rational number is a number.

Rational Numbers. The most common way to study rational numbers is to study them all at one time. Let’s begin. A rational number is a number which can be expressed as a ratio of two integers, $a/b$ where $b \neq 0$. Let $\mathbb{Q}$ denote the set of all rational numbers. That is,

\[ \mathbb{Q} = \{x \mid x = a/b, a, b \in \mathbb{Z}, b \neq 0\}, \]

where $\mathbb{Z}$ denoted the set of integers. The following exercises will help you understand the structure of $\mathbb{Q}$.

1. Prove that the set $\mathbb{Q}$ is closed under addition. That is, prove that for any two rational numbers $x = a/b$ and $y = c/d$, $x + y$ is a rational number.

Solution: We simply need to write $x + y$ as a ratio of two integers. Because
of the way we add fractions,

\[ x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} = \frac{ad}{bd} + \frac{bc}{db} = \frac{ad + bc}{bd}. \]

But \(ad + bc\) and \(bd\) are both integers because the integers are closed under both + and \(\times\).

2. Prove that the set \(\mathbb{Q}\) is closed under multiplication. That is, prove that for any two rational numbers \(x = a/b\) and \(y = c/d\), \(x \cdot y\) is a rational number.

**Solution:** We simply need to write \(x + y\) as a ratio of two integers. Because of the way we multiply fractions,

\[ x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \]

Of course, \(ac\) and \(bd\) are both integers because the integers are closed under \(\times\).

3. Prove that the number midway between two rational numbers is rational.

**Solution:** Let \(x = a/b\) and \(y = c/d\). The midpoint of \(x\) and \(y\) is

\[ \frac{x + y}{2} = \frac{ad + bc}{2bd}, \]

which is a quotient of two integers.

4. For this essay, we assume the set \(\mathbb{R}\) of real numbers is the set of all positive and negative decimal numbers. These decimals have three forms, those that terminate, ie. have only finitely many non-zero digits, like 1.12500...; those that repeat like 1.3333... = 4/3, and those that do not repeat. Prove that all rational numbers of one of the first two types, and vice-versa, any number of the first two types is rational.

**Solution:** To see that each rational number has either a terminating or repeating decimal representation, suppose \(x = a/b\) is rational with \(a\) and \(b\) integers. Dividing \(a\) by \(b\) using long division, results in a sequence of remainders, and each of these remainders is between 0 and \(b - 1\). If the remainder is ever zero,
the division process terminates and the resulting decimal has only a finite number of non-zero digits. If not, then we eventually get a repeat remainder, and once this happens, the remainders reoccur in blocks. The proof in the other direction requires more care. See the essay on representation for this proof.

5. Let \( z \) be a positive irrational number. Prove that there is a positive rational \( r \) number less than \( z \).

**Solution:** Since \( z \) is positive, we can write \( z = x_0.x_1x_2\ldots \), where either \( x_0 \) or one of the \( x_i \) is not zero. Let \( k \) denote the smallest index such that \( x_k \) is not zero. Then \( x_0.x_1x_2\ldots x_k \) is a rational number less than \( z \).

6. Prove that the rational numbers \( \mathbb{Q} \) is dense in the set of real numbers \( \mathbb{R} \). That is, prove that between any two real numbers, there is a rational number.

**Solution:** Suppose \( x \) an \( y \) are any two real numbers with \( x < y \). Let \( x = x_0.x_1x_2\ldots \), and let \( y = y_0.y_1y_2\ldots \). Suppose \( k \) is the smallest subscript where they differ. Then \( y_0.y_1y_2\ldots y_k \) is a rational number between \( x \) and \( y \).

In the following problems, we need the notion of **unit fraction**. A unit fraction is a fraction of the form \( 1/n \) where \( n \) is a positive integer. Thus, the unit fractions are \( 1/1, 1/2, 1/3, \ldots \).

1. **Fractions as Addresses** Divide the unit interval into \( n \)-ths and also into \( m \)-ths for selected, not too large, choices of \( n \) and \( m \), and then find the lengths of all the resulting subintervals. For example, for \( n = 2, m = 3 \), you get \( 1/3, 1/6, 1/6/1/3 \). For \( n = 3, m = 4 \), you get \( 1/4, 1/12, 1/6, 1/6, 1/12, 1/3 \). Try this for \( n = 3 \) and \( m = 5 \). Can you find a finer subdivision into equal intervals that incorporates all the division points for both denominators? I got this problem from Roger Howe.

**Solution:** It is probably best to do after displaying and ascertaining some appreciation for the way the multiples of a fixed unit fraction divide the number ray into equal intervals. In contrast, when you superimpose two such divisions, the resulting intervals will be quite different in length, and the situation may seem chaotic. A take away would be, you can, and the LCM of the denominators will give such. Also, there will always be at least one interval whose length is \( 1/\text{LCM} \), and of course, all intervals will be multiples of \( 1/\text{LCM} \).

2. Here’s a problem from *Train Your Brain*, by George Grätzer. ‘It is difficult to subtract fractions in your head’, said John. ‘That’s right’ said Peter, ‘but
you know. There are several tricks that can help you. You often get fractions whose numerators are one less than their denominators, for instance,

\[
\frac{1}{2} - \frac{3}{4}
\]

It’s easy to figure out the difference between two such fractions.

\[
\frac{1}{2} - \frac{3}{4} = \frac{4}{4 \times 2} = \frac{1}{4}
\]

‘Simple, right?’ Can you always use this method?

**Solution:** Yes, this always works. Let \( \frac{a}{a+1} \) and \( \frac{b}{b+1} \) be two such fractions with \( a < b \). The

\[
\frac{b}{b+1} - \frac{a}{a+1} = \frac{(a+1)b - (b+1)a}{(b+1)(a+1)} = \frac{b + 1 - (a + 1)}{(b + 1)(a + 1)}.
\]

3. Show that every unit fraction can be expressed as the sum of two different unit fractions.

**Solution:** Note that \( \frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \).

4. **Sums of unit fractions**

(a) Notice that \( \frac{2}{7} \) is expressible as the sum of two unit fractions: \( \frac{2}{7} = \frac{1}{4} + \frac{1}{28} \). But \( \frac{3}{7} \) cannot be so expressed. Show that \( \frac{3}{7} \) is not the sum of two unit fractions.

(b) There is a conjecture of Erdős that every fraction \( \frac{4}{n} \) where \( n \geq 3 \) can be written as the sum of three unit fractions with different denominators. Verify the Erdős conjecture for \( n = 23, 24, \) and \( 25 \).

**Solution:** \( \frac{4}{25} = \frac{1}{8} + \frac{1}{40} + \frac{1}{100} \).

(c) Can you write \( 1 \) as a sum of different unit fractions all with odd denominators?

**Solution:** Yes, it can be done. One way is to add the reciprocals of each of the following: \( 3, 5, 7, 9, 11, 15, 21, 165, 693 \). I suspect there are lots of other ways also.

(d) Can any rational number \( r, 0 < r < 1 \) be represented as a sum of unit fractions?

**Solution:** You just use the greedy algorithm: subtract the largest unit fraction less than your current fraction. You can show that the numerator of the resulting fraction is always less than the numerator you started with, so the process converges.
5. **In the Space Between**

   (a) Name a fraction between 1/2 and 2/3. Give an argument that your fraction satisfies the condition.

   (b) Name a fraction between 11/15 and 7/10. How about 6/7 and 11/13?

   (c) Name the fraction with smallest denominator between 11/15 and 7/10. Or 6/7 and 11/13?

   (d) First draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into 7 + 13 = 20 equal pieces. How many marks are on each piece?

   (e) A bicycle team of 7 people brings 6 water bottles, while another team of 13 people brings 11 water bottles. What happens when they share? Some of this material is from Josh Zucker’s notes on fractions taken from a workshop for teachers at American Institute of Mathematics, summer 2009. Some of the material is from the book Algebra by Gelfand and Shen.

6. **Dividing Horses**

   This problem comes from *Dude, Can You Count*, by Christian Constanda. An old cowboy dies and his three sons are called to the attorney’s office for the reading of the will.

   All I have in this world I leave to my three sons, and all I have is just a few horses. To my oldest son, who has been a great help to me and done a lot of hard work, I bequeath half my horses. To my second son, who has also been helpful but worked a little less, I bequeath a third of my horses, and to my youngest son, who likes drinking and womanizing and hasn’t helped me one bit, I leave one ninth of my horses. This is my last will and testament.

   The sons go back to the corral and count the horses, wanting to divide them according to their pa’s exact wishes. But they run into trouble right away when they see that there are 17 horses in all and that they cannot do a proper division. The oldest son, who is entitled to half—that is 8½ horses—wants to take 9. His brothers immediately protest and say that he cannot take more than that which he is entitled to, even if it means calling the butcher. Just as they are about to have a fight, a stranger rides up and agrees to help. They explain to him the problem. Then the stranger dismounts, lets his horse mingle with the others, and says “Now there are 18 horses in the corral, which is a
much better number to split up. Your share is half” he says to the oldest son, “and your share is six”, he says to the second. “Now the third son can have one ninth of 18, which is two, and there is $18 - 9 - 6 - 2 = 1$ left over. The stranger gets on the $18$th horse and rides away. How was this kind of division possible.

**Solution:** The sum of the three fractions is less than 1: $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$. So the stranger’s horse helps complete the whole.

7. Consider the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{5}{12}.$$  

Find all the ordered pairs $(a, b)$ of real number solutions.

**Solution:** Thanks to Randy Harter for this problem. First, let’s try to find all integer solutions. To that end, rewrite the equation as

$$12(a + b) = 5ab.$$  

Thus

$$0 = 5ab - 12b - 12a$$

$$= 5ab - 12b - 12a + \frac{144}{5} - \frac{144}{5}$$

$$= b(5a - 12) - \frac{12}{5} (5a - 12) - \frac{144}{5}$$

$$= (5a - 12)(5b - 12) - 144$$

Now, factoring $144 = 2^43^2$ and looking at pairs of factors, we can find all the integral solutions. What about the rest. To this end, replace $a$ with $x$ and $b$ with $f(x)$ and solve for $f(x)$. We get

$$f(x) = \left(\frac{5}{12} - \frac{1}{x}\right)^{-1} = \frac{12x}{5x - 12}.$$  

This rational function has a single vertical asymptote at $x = 12/5$ and a zero at $x = 0$. So you can see that for each $0 \neq x \neq 12/5$, there is a $y$ such that $1/x + 1/y = 5/12$.

8. Suppose $\{a, b, c, d\} = \{1, 2, 3, 4\}$. 
(a) What is the smallest possible value of \( a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}} \).

**Solution:** We can minimize the value by making the integer part as small as possible and then making the denominators as large as possible. Clearly \( a = 1 \) is the best we can do, and then \( b = 4 \) is certainly best, etc. So we get

\[
1 + \frac{1}{4 + \frac{1}{2 + \frac{1}{3}}} = \frac{38}{31}.
\]

(b) What is the largest possible value of \( a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}} \).

**Solution:** We can maximize the value by making the integer part as large as possible and then making the denominators as small as possible. Clearly \( a = 4 \) is the best we can do, and then \( b = 1 \) is certainly best, etc. So we get

\[
4 + \frac{1}{1 + \frac{1}{3 + \frac{1}{2}}} = \frac{43}{9}.
\]

9. **Smallest Sum.** Using each of the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try \( 99/96 + 97/98 \) but that is not the smallest sum. This problem is due to Sam Vandervelt. Extend this problem as follows. Suppose \( 0 < a < b < c < d \) are all integers. What is the smallest possible sum of two fractions that use each integer as a numerator or denominator? What is the largest such sum? What is we have six integers, \( 0 < a < b < c < d < e < f \). Now here’s a sequence of easier problems that might help answer the ones above.

(a) How many fractions \( a/b \) can be built with \( a, b \in \{1, 2, 3, 4\} \), and \( a \neq b \)?

**Solution:** There are 12 fractions.

(b) How many of the fractions in (a) are less than 1?

**Solution:** There are 6 in this set, \( 1/2, 1/3, 1/4, 2/3, 2/4, 3/4 \).

(c) What is the smallest number of the form \( \frac{a}{b} + \frac{c}{d} \), where \( \{a, b, c, d\} = \{1, 2, 3, 4\} \)?

**Solution:** There are just three pairs of fractions to consider, \( A = \frac{1}{3} + \frac{2}{4} \), \( B = \frac{1}{4} + \frac{2}{3} \), and \( C = \frac{1}{2} + \frac{3}{4} \). Why is \( C \) not a candidate for the least
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value? Note that $A$ beats $B$ here because $A = \frac{1}{3} + \frac{2}{4} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4}$ while $B = \frac{1}{4} + \frac{2}{3} = \frac{1}{4} + \frac{1}{3} + \frac{1}{3}$. Does this reasoning work for any four positive integers $a < b < c < d$? The argument for four arbitrary positive integers is about the same. Let $A = \frac{a}{c} + \frac{b}{d}$, $B = \frac{a}{d} + \frac{b}{c}$. Then $A = \frac{a}{c} + \frac{a}{d} + \frac{b-a}{d}$, while $B = \frac{a}{c} + \frac{a}{d} + \frac{b-a}{c}$, and since $c < d$, $A < B$.

(d) What is the largest number of the form $\frac{a}{b} + \frac{c}{d}$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$?

Solution: Again there are just three pairs of fractions to consider, $A = \frac{3}{1} + \frac{1}{2}$, $B = \frac{4}{1} + \frac{1}{2}$, and $C = \frac{2}{1} + \frac{1}{2}$. Why is $C$ not a candidate for the least value? Note that $B$ beats $A$. Does this reasoning work for any four positive integers $a < b < c < d$?

10. **Simpson's** (with thanks to http://www.cut-the-knot.com) Bart and Lisa shoot free throws in two practice sessions to see who gets to start in tonight’s game. Bart makes 5 out of 11 in the first session while Lisa makes 3 out of 7. Who has the better percentage? Is it possible that Bart shoots the better percentage again in the second session, yet overall Lisa has a higher percentage of made free throws? The answer is yes! This phenomenon is called Simpson’s Paradox.

(a) Find a pair of fractions $a/b$ for Bart and $k/l$ for Lisa such that $a/b > k/l$ and yet, Lisa’s percentage overall is better.

Solution: One solution is that Bart makes 6 out of 9 in the second session while Lisa makes 9 out of 14. Now $12/21 > 11/20$. The numbers $12/21$ and $11/20$ are called mediants. Specifically, given two fractions $a/b$ and $c/d$, where all of $a, b, c$, and $d$ are positive integers, the mediant of $a/b$ and $c/d$ is the fraction $(a + c)/(b + d)$.

(b) Why is the mediant of two fractions always between them?

Solution: Imagine two groups of bicyclists, one with $b$ riders and $a$ bottles of water among them, and the other with $d$ riders and $c$ bottles of water among them. When they meet, they all agree to share equally. Thus we have $b + d$ cyclists with a total of $a + c$ bottles of water. Is it clear that $\frac{a+c}{b+d}$ lies between $a/b$ and $c/d$?

(c) Notice that the mediant of two fractions depends on the way they are represented and not just on their value. Explain Simpson’s Paradox in terms of mediants.

(d) Define the mediant $M$ of two fractions $a/b$ and $c/d$ with the notation $M(a/b, c/d)$. So $M(a/b, c/d) = (a + c)/(b + d)$. This operation is sometimes called ‘student addition’ because many students think this would be a good way to add fractions. Compute the mediants $M(1/3, 8/9)$ and
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$M(4/9, 2/2)$ and compare each mediant with the midpoint of the two fractions.

Now let's see what the paradox looks like geometrically on the number line. Here, $B_1$ and $B_2$ represent Bart's fractions, $L_1, L_2$ Lisa's fractions, and $M_B, M_L$ the two mediants.

\[
\begin{array}{cccccc}
L_1 & B_1 & M_B & M_L & L_2 & B_2 \\
\end{array}
\]

(e) (Bart wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $a/b < s/t < 1/2$

ii. $c/d < u/v < 1$, and

iii. $\frac{s+u}{t+u} < \frac{a+c}{b+d}$

(f) (Lisa wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $s/t < a/b < 1/2$

ii. $c/d < u/v < 1$, and

iii. $\frac{s+u}{t+u} < \frac{a+c}{b+d}$

11. Fabulous Fractions

(a) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to fill in the boxes so that the arithmetic is correct.

\[
\begin{array}{ccc}
\square & \times & \square \\
\square & \times & \square \\
\square & \times & \square \\
\end{array}\quad + \quad \begin{array}{ccc}
\square & \times & \square \\
\square & \times & \square \\
\square & \times & \square \\
\end{array} = 1.
\]

What is the largest of the three fractions?

**Solution:** We may assume that the second numerator is 5 and the third is 7. If either 5 or 7 is used in a denominator, it can never be neutralized. Since the least common multiple of the remaining numbers is 72, we can use 1/72 as the unit of measurement. Now one of the three fractions must be close to 1. This can only be $5/(2 \cdot 3)$ or $7/(2 \cdot 4)$. In the first case, we are short 12 units. Of this, 7 must come from the third fraction so that the other 5 must come from the first fraction. This is impossible because the first fraction has numerator 1 and 5 does not divide 72. In the second
In case, we are 9 units short. If this, 5 must come from the second fraction and 4 must come from the third. This can be achieved as shown below.

\[
\frac{1}{3 \times 6} + \frac{5}{8 \times 9} + \frac{7}{2 \times 4} = 1.
\]

(b) Find four different decimal digits \(a, b, c, d\) so that \(\frac{a}{b} + \frac{c}{d} < 1\) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

**Solution:** \(\frac{7}{8} + \frac{1}{9} = \frac{71}{72}\). This is as large as can be. Why?

(c) Next find six different decimal digits \(a, b, c, d, e, f\) so that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1\) and the sum is as large as possible.

**Solution:** One is \(\frac{7}{9} + \frac{1}{8} + \frac{2}{9} = \frac{503}{504}\).

(d) Find four different decimal digits \(a, b, c, d\) so that \(\frac{a}{b} + \frac{c}{d} < 2\) but is otherwise as large as possible. Prove that your answer is correct. Then change the 2 to 3 and to 4.

**Solution:** The denominator cannot be 72 or 63. Why? Trying for a fraction of the form \(\frac{2n-1}{n}\), where \(n = 56\), we are lead to \(\frac{9}{8} + \frac{9}{7} = \frac{111}{56}\). Why?

(e) Next find six different decimal digits \(a, b, c, d, e, f\) so that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 2\) and the sum is as large as possible. Then change the 2 to 3 and to 4.

**Solution:** Finally find four different decimal digits \(a, b, c, d\) so that \(\frac{a}{b} + \frac{c}{d} > 1\) but is otherwise as small as possible. Prove that your answer is correct. Then change the 1 to 2 and to 3.

**Solution:**

12. Use exactly eight digits to form four two digit numbers \(ab, cd, ed, gh\) so that the sum \(\frac{ab}{cd} + \frac{ef}{gh}\) is as small as possible. As usual, interpret \(ab\) as \(10a + b\), etc.

**Solution:** The answer is \(\frac{13}{34} + \frac{24}{96}\). First, the four numerator digits are 1, 2, 3, 4 and the four denominator digits are 6, 7, 8, and 9. Also, if \(\frac{ab}{cd} + \frac{ef}{gh}\) is as small as possible, then \(a < b\) and \(c > d, e < f\) and \(g > h\). For convenience, we assume \(c < g\). Then \(\frac{13}{34} + \frac{24}{96} = \frac{34}{68} + \frac{16}{96} > \frac{13}{34} + \frac{24}{96}\) because \(\frac{10}{96} > \frac{1}{34}\). Now compare \(\frac{13}{34} + \frac{24}{96}\) with \(\frac{13}{34} + \frac{24}{96}\). Pretty clearly, \(\frac{13}{34} - \frac{13}{34} = 13\left(\frac{1}{34} - \frac{1}{96}\right) > 24\left(\frac{1}{96} - \frac{1}{96}\right) = \frac{24}{96}\).

13. Next find six different decimal digits \(a, b, c, d, e, f\) so that \(\frac{a}{b} + \frac{c}{d} = \frac{e}{f}\).

**Solution:** There are many solutions. One is \(\frac{1}{3} + \frac{2}{4} = \frac{5}{6}\).
14. **Problems with Four Fractions.** These problems can be very tedious, with lots of checking required. They are not recommended for children.

(a) For each \(i = 1, 2, \ldots, 9\), use all the digits except \(i\) to solve the equation

\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N
\]

for some integer \(N\). In other words arrange the eight digits so that the sum of the four fractions is a whole number. For example, when \(i = 8\) we can write

\[
\frac{9}{1} + \frac{5}{2} + \frac{4}{3} + \frac{7}{6} = 14.
\]

(b) What is the smallest integer \(k\) such that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k\)? Which digit is left out?

**Solution:** The smallest achievable \(k\) is 2. It can be achieved when the digit left out is \(i = 5\) or \(i = 7\): 

\[
\frac{1}{4} + \frac{7}{6} + \frac{2}{8} + \frac{3}{9} = 5/4 + 1/6 + 2/8 + 3/9 = 2.
\]

(c) What is the largest integer \(k\) such that \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k\)? Which digit is left out?

**Solution:** The largest achievable \(k\) is 16. It can be achieved when the digit left out is \(i = 4\) or \(i = 5\): 

\[
\frac{9}{1} + \frac{7}{2} + \frac{8}{3} + \frac{5}{6} = 8/1 + 7/2 + 9/3 + 6/4 = 9/1 + 8/2 + 7/3 + 4/6 = 16.
\]

(d) For what \(i\) do we get the greatest number of integers \(N\) for which \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N\), where \(S_i = \{a, b, c, d, e, f, g, h\}\)?

**Solution:** For \(i = 5\) there are 36 solutions. For \(i = 1, \ldots, 9\) we have 4, 3, 5, 4, 36, 3, 25, 7, and 21 solutions respectively.

(e) Consider the fractional part of the fractions. Each solution of \(\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N\) belongs to a class of solutions with the same set of fractional parts. For example, \(5/2 + 8/4 + 7/6 + 3/9 = 6\) and \(5/1 + 7/6 + 4/8 + 3/9 = 7\) both have fractional parts sets \(\{1/2, 1/3, 1/6\}\). How many different fractional parts multisets are there?

**Solution:** There are 11 sets of fractional parts including the empty set.

They are \(\phi, \{1/2, 1/2\}, \{1/4, 3/4\}, \{1/3, 2/3\}, \{1/4, 1/4, 1/2\}, \{1/2, 3/4, 3/4\}, \{1/2, 1/3, 1/6\}, \{1/2, 2/3, 5/6\}, \{1/3, 1/2, 1/2, 2/3\}, \{1/6, 1/4, 1/4, 1/3\}, \{1/4, 1/3, 2/3, 3/4\}\).

(f) Find all solutions to

\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = i
\]

where each letter represents a different nonzero digit.
Solution: There are two solutions with $i = 5$, one with $i = 7$, one with $i = 8$ and four with $i = 9$.

(g) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 2i \]
where each letter represents a different nonzero digit.

Solution: There are two solutions with $i = 5$, one with $i = 6$, and three with $i = 7$.

(h) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 3i \]
where each letter represents a different nonzero digit.

Solution: There is one solution with $i = 1$, one with $i = 4$, and four with $i = 5$.

(i) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 4i \]
where each letter represents a different nonzero digit.

Solution: There are two solutions with $i = 3$ and one with $i = 4$.

(j) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i \]
where each letter represents a different nonzero digit.

Solution: There is only one solution: $5/1 + 7/3 + 8/4 + 6/9 = 5 \cdot 2 = 10$.

(k) Find the maximum integer value of
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i \]
where each letter represents a different nonzero digit.

Solution: Let $G_i$ denote the largest integer value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i$ where each letter represents a different nonzero digit. The largest possible value is 12. It is achieved when $i = 4$; $9/1 + 7/2 + 8/3 + 5/6 - 4 = 12$. The reasoning goes like this. The largest possible value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}$ is $9/1 + 8/2 + 7/3 + 6/4 = 16 + 2/3$, so the largest possible integer value is 16. This means that we need only check the values $G_1, G_2,$ and $G_3$. There are only four ways to get an integer by arranging the digits 2, 3, 4, 5, 6, 7, 8, 9 in the form $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}$, and the resulting integers are 3, 6, 6, and 8. So
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= 8 − 1 = 7. In case i = 2 there are just three ways and the integers 7, 10, and 12 result, so \( G_2 = 12 − 2 = 10 \). Finally, \( G_3 \) is the largest of 10 − 3, 12 − 3, 13 − 3, and 14 − 3 = 11. It would be enough to show that we cannot achieve 16 using all the nonzero digits except 3, 15 without the 2, or 14 without the 1, and this does not take too much work.

15. **Ford Circles.** Let \( r = a/b \) be a rational number where \( a \) and \( b \) are positive integers with no common factors. The Ford circle \( F_r \) is a circle tangent to the \( x \)-axis at the point \( r \) with diameter \( 1/b^2 \). For example \( F_{1/2} \) has the equation \((x - \frac{1}{2})^2 + (y - \frac{1}{4})^2 = \frac{1}{64}\).

(a) Find an equation for the Ford circles \( F_1 \) and \( F_{2/3} \).

**Solution:** \( F_1 \) is given by \((x - 1)^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\) and \( F_{2/3} \) is given by \((x - \frac{2}{3})^2 + (y - \frac{1}{18})^2 = \frac{1}{18^2}\).
Shown here are the Ford circles $F_1, F_\frac{1}{2}$ and $F_\frac{2}{3}$.

(b) Drawing the Ford circles $F_1$ and $F_\frac{1}{2}$, it appears that they are tangent to one another. Prove it.

(c) Next, construct the Ford circle $F_\frac{2}{3}$, and notice that it appears to be tangent to both $F_1$ and $F_\frac{1}{2}$. Prove that it is.

(d) Now consider two arbitrary Ford circles $F_\frac{r}{s}$ and $F_\frac{p}{q}$, where $\frac{r}{s}$ is close to
\[ \frac{p}{q} \text{ in the sense that } |ps - qr| = 1. \text{ Show that the two Ford circles are tangent.} \]

16. **Farey Sequences.** A Farey sequence of order \( n \) is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to \( n \), arranged in order of increasing size.

Each Farey sequence starts with the value 0, denoted by the fraction 0/1, and ends with the value 1, denoted by the fraction 1/1 (although some authors omit these terms). So \( F_1 = \{0/1,1/1\} \), \( F_2 = \{0/1,1/2,1/1\} \), and \( F_3 = \{0/1,1/3,1/2,2/3,1/1\} \).

(a) Build the sequences \( F_4 \), \( F_5 \) and \( F_6 \).

(b) How is each Farey sequence related to the one that came just before it?