$$q = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{3} = 7$$

$$|7 - \lambda_q| = 2.4495$$

8. Consider the matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix}$ and the vector $(1,1,1)^T$. Find the shortest distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

$$q = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{3} = 8.0$$

$$\begin{aligned} |\lambda_q - 8| &\leq \frac{\left| \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 8 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|}{\sqrt{3}} \\ &= 3.2660 \end{aligned}$$

9. Using Gerschgorin's theorem, find upper and lower bounds for the eigenvalues of $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & -3 \end{pmatrix}$.

$-10 \leq \lambda \leq 12$

10. Tell how to find a matrix whose characteristic polynomial is a given monic polynomial. This is called a companion matrix. Find the roots of the polynomial $x^3 + 7x^2 + 3x + 7$.

The companion matrix is $\begin{pmatrix} 0 & 0 & -7 \\ 1 & 0 & -3 \\ 0 & 1 & -7 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & -7 \\ 1 & 0 & -3 \\ 0 & 1 & -7 \end{pmatrix}^{20} = \begin{pmatrix} 8.0623 \times 10^{14} & -5.4085 \times 10^{15} & 3.6282 \times 10^{16} \\ 2.2535 \times 10^{14} & -1.5117 \times 10^{15} & 1.0141 \times 10^{16} \\ 7.7264 \times 10^{14} & -5.1832 \times 10^{15} & 3.477 \times 10^{16} \end{pmatrix}$$

$$= \begin{pmatrix} 0.70772 & 0.25858 & 0.65747 \\ 0.19782 & 0.82086 & -0.53578 \\ 0.67823 & -0.50924 & -0.52979 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1.1392 \times 10^{15} & -7.6422 \times 10^{15} & 5.1266 \times 10^{16} \\ 0 & 4.5704 \times 10^{10} & 2.0791 \times 10^{10} \\ 0 & 0 & 3.1514 \times 10^{11} \end{pmatrix}$$

$$\begin{pmatrix} 0.70772 & 0.25858 & 0.65747 \\ 0.19782 & 0.82086 & -0.53578 \\ 0.67823 & -0.50924 & -0.52979 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & -7 \\ 1 & 0 & -3 \\ 0 & 1 & -7 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 0.70772 & 0.25858 & 0.65747 \\ 0.19782 & 0.82086 & -0.53578 \\ 0.67823 & -0.50924 & -0.52979 \end{pmatrix}$$

$$= \begin{pmatrix} -6.7083 & 5.8506 & 5.2209 \\ 4.1472\times 10^{-5} & 0.15476 & 1.1876 \\ -1.3916\times 10^{-5} & -0.93681 & -0.44646 \end{pmatrix}$$

Clearly a real root is close to -6.7083. Then the other roots can be obtained from the lower right block.

$\begin{pmatrix} 0.15476 & 1.1876 \\ -0.93681 & -0.44646 \end{pmatrix}$, eigenvalues: $-0.14585 + 1.011i, -0.14585 - 1.011i$

How well do these work? Try the last one. Evaluating the polynomial at this value of x gives

$6.3505\times 10^{-4} + 2.0658\times 10^{-4}i$

11. Find the roots to $x^4 + 3x^3 + 4x^2 + x + 1$. It has two complex roots.

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}^{20} =$$

$$\begin{pmatrix} 36482. & -27300.0 & -49899. & 2.4489\times 10^{5} \\ 14010.0 & 9182.0 & -77199. & 1.9499\times 10^{5} \\ 1.3180\times 10^{5} & -95190.0 & -1.9041\times 10^{5} & 9.0235\times 10^{5} \\ 27300.0 & 49899. & -2.4489\times 10^{5} & 5.4425\times 10^{5} \end{pmatrix} =$$

$$\begin{pmatrix} 0.26030 & -6.9651\times 10^{-2} & -0.94185 & 0.20079 \\ 9.9960\times 10^{-2} & 0.25172 & 0.20932 & 0.93959 \\ 0.94038 & -0.20295 & 0.25314 & -0.10207 \\ 0.19478 & 0.94371 & -7.0906\times 10^{-2} & -0.25775 \end{pmatrix}$$

$$\begin{pmatrix} 1.4016\times 10^{5} & -85984. & -2.4746\times 10^{5} & 1.0378\times 10^{6} \\ 0 & 70622. & -2.0842\times 10^{5} & 3.6251\times 10^{5} \\ 0 & 0 & 2.1305 & -5.038 \\ 0 & 0 & 0 & 2.6834 \end{pmatrix}$$

$$\begin{pmatrix} 0.26030 & -6.9651\times 10^{-2} & -0.94185 & 0.20079 \\ 9.9960\times 10^{-2} & 0.25172 & 0.20932 & 0.93959 \\ 0.94038 & -0.20295 & 0.25314 & -0.10207 \\ 0.19478 & 0.94371 & -7.0906\times 10^{-2} & -0.25775 \end{pmatrix}^{T}$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 0.26030 & -6.9651\times 10^{-2} & -0.94185 & 0.20079 \\ 9.9960\times 10^{-2} & 0.25172 & 0.20932 & 0.93959 \\ 0.94038 & -0.20295 & 0.25314 & -0.10207 \\ 0.19478 & 0.94371 & -7.0906\times 10^{-2} & -0.25775 \end{pmatrix} =$$

$$\begin{pmatrix} -0.61347 & -4.2510 & 0.48569 & 2.0968 \\ 0.50389 & -2.3376 & 0.11542 & 0.33094 \\ 2.5476\times 10^{-7} & 8.4187\times 10^{-6} & -0.15734 & 0.30446 \\ 4.6118\times 10^{-6} & 7.5855\times 10^{-6} & -0.97448 & 0.10846 \end{pmatrix}$$

Now there are two blocks of importance.

$\begin{pmatrix} -0.61347 & -4.2510 \\ 0.50389 & -2.3376 \end{pmatrix}$, eigenvalues: $-1.4755 + 1.1827i, -1.4755 - 1.1827i$

$\begin{pmatrix} -0.157\,34 & 0.304\,46 \\ -0.974\,48 & 0.108\,46 \end{pmatrix}$, eigenvalues: $-0.024\,44 + 0.528\,23i, -0.024\,44 - 0.528\,23i$

How well does one of these work? Try one.

$(-0.024\,44 + 0.528\,23i)^4 + 3\,(-0.024\,44 + 0.528\,23i)^3$

$+4\,(-0.024\,44 + 0.528\,23i)^2 + (-0.024\,44 + 0.528\,23i)$

$+1 = 2.\,887\,9 \times 10^{-5} - 3.\,009\,2 \times 10^{-6}i$

12. Suppose A is a real symmetric matrix and the technique of reducing to an upper Hessenberg matrix is followed. Show the resulting upper Hessenberg matrix is actually equal to 0 on the top as well as the bottom.

    Let $Q^T AQ = H$ where H is upper Hessenberg. Then take the transpose of both sides. This will show that $H = H^T$ and so H is zero on the top as well.