

## **Primer for Ordinary Differential Equations**

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*After reading this chapter, you should be able to:*

1. *define an ordinary differential equation,*
2. *differentiate between an ordinary and partial differential equation, and*
3. *solve linear ordinary differential equations with fixed constants by using classical solution and Laplace transform techniques.*

### **Introduction**

An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems leads to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them.

Differential equations are of two types

- (A) ordinary differential equations (ODE)
- (B) partial differential equations (PDE)

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable. Examples of ordinary differential equations include

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0, \quad \frac{dy}{dx}(0) = 2, \quad y(0) = 4,$$

$$\frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + y = \sin x, \quad \frac{d^2 y}{dx^2}(0) = 12, \quad \frac{dy}{dx}(0) = 2, \quad y(0) = 4$$

Ordinary differential equations are classified in terms of order and degree. *Order* of an ordinary differential equation is the same as the highest derivative and the *degree* of an ordinary differential equation is the power of highest derivative.

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Thus the differential equation,

$$x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + xy = e^x$$

is of order 3 and degree 1, whereas the differential equation

$$\left( \frac{dy}{dx} + 1 \right)^2 + x^2 \frac{dy}{dx} = \sin x$$

is of order 1 and degree 2.

An engineer's approach to differential equations is different from a mathematician.

While, the latter is interested in the mathematical solution, an engineer should be able to interpret the result physically. So, an engineer's approach can be divided into three phases:

- a) formulation of a differential equation from a given physical situation,
- b) solving the differential equation and evaluating the constants, using given conditions, and
- c) interpreting the results physically for implementation.

### **Formulation of differential equations**

As discussed above, the formulation of a differential equation is based on a given physical situation. This can be illustrated by a spring-mass-damper system.

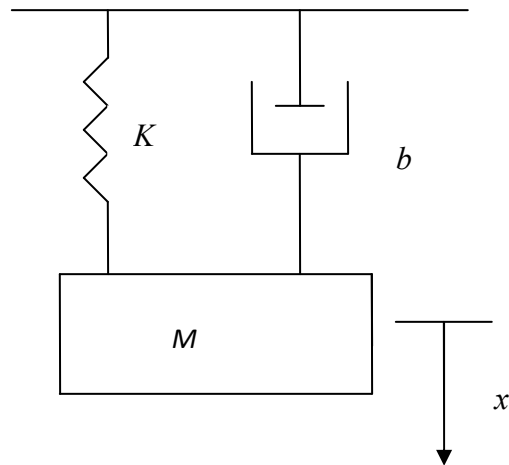
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**Figure 1** Spring-mass damper system.

Above is the schematic diagram of a spring-mass-damper system. A block is suspended freely using a spring. As most physical systems involve some kind of damping - viscous damping, dry damping, magnetic damping, etc., a damper or dashpot is attached to account for viscous damping.

Let the mass of the block be  $M$ , the spring constant be  $K$ , and the damper coefficient be  $b$ . If we measure displacement from the static equilibrium position we need not consider gravitational force as it is balanced by tension in the spring at equilibrium.

Below is the free body diagram of the block at static and dynamic equilibrium. So, the equation of motion is given by

$$Ma = F_S + F_D \quad (1)$$

where

$F_S$  is the restoring force due to spring.

$F_D$  is the damping force due to the damper.

$a$  is the acceleration.

The restoring force in the spring is given by



$$F_s = -Kx \quad (2)$$

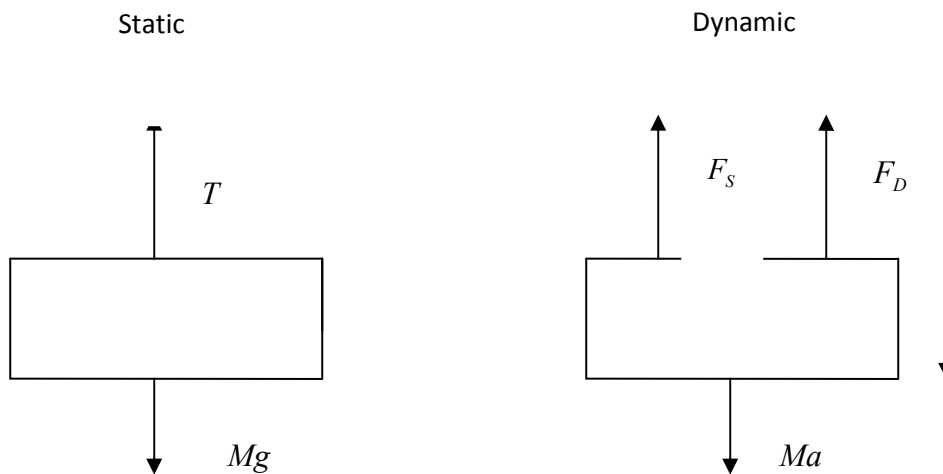
as the restoring force is proportional to displacement and it is negative as it opposes the motion. The damping force in the damper is given by

$$F_D = -bv \quad (3)$$

as the damping force is directly proportional to velocity and also opposes motion.

Therefore, the equation of motion can be written as

$$Ma = -Kx - bv \quad (4)$$



**Figure 2** Free body diagram of spring-mass-damper system.

Since

$$a = \frac{d^2x}{dt^2} \text{ and } v = \frac{dx}{dt}$$

from Equation (4), we get



$$M \frac{d^2 x}{dt^2} = -Kx - b \frac{dx}{dt}$$

$$M \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + Kx = 0 \tag{5}$$

This is an ordinary differential equation of second order and of degree one.

### Solution to linear ordinary differential equations

In this section we discuss two techniques used to solve ordinary differential equations

- (A) Classical technique
- (B) Laplace transform technique

#### Classical Technique

The general form of a linear ordinary differential equation with constant coefficients is given by

$$\frac{d^n y}{dx^n} + k_n \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_3 \frac{d^2 y}{dx^2} + k_2 \frac{dy}{dx} + k_1 y = F(x) \tag{6}$$

The general solution contains two parts

$$y = y_H + y_P \tag{7}$$

where

$y_H$  is the homogeneous part of the solution and

$y_P$  is the particular part of the solution.

The homogeneous part of the solution  $y_H$  is that part of the solution that gives zero when substituted in the left hand side of the equation. So,  $y_H$  is solution of the equation

$$\frac{d^n y}{dx^n} + k_n \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_3 \frac{d^2 y}{dx^2} + k_2 \frac{dy}{dx} + k_1 y = 0 \tag{8}$$



The above equation can be symbolically written as

$$D^n y + k_n D^{n-1} y + \dots + k_2 D y + k_1 y = 0 \tag{9}$$

$$(D^n + k_n D^{n-1} + \dots + k_2 D + k_1) y = 0 \tag{10}$$

where,

$$D^n = \frac{d^n}{dx^n} \tag{11}$$

$$D^{n-1} = \frac{d^{n-1}}{dx^{n-1}}$$

.  
.
   
.

operating on  $y$  is the same as

$$(D - r_1), (D - r_2), (D - r_n)$$

operating one after the other in any order, where

$$(D - r_1), (D - r_2), \dots, (D - r_n)$$

are factors of

$$D^n + k_n D^{n-1} + \dots + k_2 D + k_1 = 0 \tag{12}$$

To illustrate

$$(D^2 - 3D + 2)y = 0$$

is same as

$$(D - 2)(D - 1)y = 0$$

$$(D - 1)(D - 2)y = 0$$



Therefore,

$$(D^n + k_n D^{n-1} + \dots + k_2 D + k_1)y = 0 \tag{13}$$

is same as

$$(D - r_n)(D - r_{n-1}) \dots (D - r_1)y = 0 \tag{14}$$

operating one after the other in any order.

Case 1: Roots are real and distinct

The entire left hand side becomes zero if  $(D - r_1)y = 0$ . Therefore, the solution to  $(D - r_1)y = 0$  is a solution to a homogeneous equation.  $(D - r_1)y = 0$  is called Leibnitz's linear differential equation of first order and its solution is

$$(D - r_1)y = 0 \tag{15}$$

$$\frac{dy}{dx} = r_1 y \tag{16}$$

$$\frac{dy}{y} = r_1 dx \tag{17}$$

Integrating both sides we get

$$\ln y = r_1 x + c \tag{18}$$

$$y = ce^{r_1 x} \tag{19}$$

Since any of the  $n$  factors can be placed before  $y$ , there are  $n$  different solutions corresponding to  $n$  different factors given by

$$C_n e^{r_n x}, C_{n-1} e^{r_{n-1} x}, \dots, C_2 e^{r_2 x}, C_1 e^{r_1 x}$$

where

$r_n, r_{n-1}, \dots, r_2, r_1$  are the roots of Equation (12) and

$C_n, C_{n-1}, \dots, C_2, C_1$  are constants.



We get the general solution for a homogeneous equation by superimposing the individual Leibnitz's solutions. Therefore

$$y_H = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots + C_{n-1} e^{r_{n-1} x} + C_n e^{r_n x} \quad (20)$$

Case 2: Roots are real and identical

If two roots of a homogeneous equation are equal, say  $r_1 = r_2$ , then

$$(D - r_n)(D - r_{n-1}) \dots (D - r_1)(D - r_1)y = 0 \quad (21)$$

Let's work at

$$(D - r_1)(D - r_1)y = 0 \quad (22)$$

If

$$(D - r_1)y = z \quad (23)$$

then

$$(D - r_1)z = 0$$

$$z = C_2 e^{r_1 x} \quad (24)$$

Now substituting the solution from Equation (24) in Equation (23)

$$(D - r_1)y = C_2 e^{r_1 x}$$

$$\frac{dy}{dx} - r_1 y = C_2 e^{r_1 x}$$

$$e^{-r_1 x} \frac{dy}{dx} - r_1 e^{-r_1 x} y = C_2$$

$$\frac{d(e^{-r_1 x} y)}{dx} = C_2$$

$$d(e^{-r_1 x} y) = C_2 dx \quad (25)$$

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Integrating both sides of Equation (25), we get

$$e^{-r_1 x} y = C_2 x + C_1$$

$$y = (C_2 x + C_1) e^{r_1 x} \quad (26)$$

Therefore the final homogeneous solution is given by

$$y_H = (C_1 + C_2 x) e^{r_1 x} + C_3 e^{r_3 x} + \dots + C_n e^{r_n x} \quad (27)$$

Similarly, if  $m$  roots are equal the solution is given by

$$y_H = (C_1 + C_2 x + C_3 x^2 + \dots + C_m x^{m-1}) e^{r_m x} + C_{m+1} e^{r_{m+1} x} + \dots + C_n e^{r_n x} \quad (28)$$

### Case 3: Roots are complex

If one pair of roots is complex, say  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$ ,

where

$$i = \sqrt{-1}$$

then

$$y_H = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} + C_3 e^{r_3 x} + \dots + C_n e^{r_n x} \quad (29)$$

Since

$$e^{i\beta x} = \cos \beta x + i \sin \beta x, \text{ and} \quad (30a)$$

$$e^{-i\beta x} = \cos \beta x - i \sin \beta x \quad (30b)$$

then

$$y_H = C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) + C_3 e^{r_3 x} + \dots + C_n e^{r_n x}$$

$$= (C_1 + C_2) e^{\alpha x} \cos \beta x + i(C_1 - C_2) e^{\alpha x} \sin \beta x + C_3 e^{r_3 x} + \dots + C_n e^{r_n x}$$

$$= e^{\alpha x} (A \cos \beta x + B \sin \beta x) + C_3 e^{r_3 x} + \dots + C_n e^{r_n x} \quad (31)$$



where

$$A = C_1 + C_2 \text{ and}$$

$$B = i(C_1 - C_2) \tag{32}$$

Now, let us look at how the particular part of the solution is found. Consider the general form of the ordinary differential equation

$$(D^n + k_n D^{n-1} + k_{n-1} D^{n-2} + \dots + k_1)y = X \tag{33}$$

The particular part of the solution  $y_p$  is that part of solution that gives  $X$  when substituted for  $y$  in the above equation, that is,

$$(D^n + k_n D^{n-1} + k_{n-1} D^{n-2} + \dots + k_1)y_p = X \tag{34}$$

### Sample Case 1

When  $X = e^{ax}$ , the particular part of the solution is of the form  $Ae^{ax}$ . We can find  $A$  by substituting  $y = Ae^{ax}$  in the left hand side of the differential equation and equating coefficients.

### **Example 1**

Solve

$$3 \frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = 5$$

### **Solution**

The homogeneous solution for the above equation is given by

$$(3D + 2)y = 0$$

The characteristic equation for the above equation is given by

$$3r + 2 = 0$$

The solution to the equation is



$$r = -0.666667$$

$$y_H = Ce^{-0.666667x}$$

The particular part of the solution is of the form  $Ae^{-x}$

$$3\frac{d(Ae^{-x})}{dx} + 2Ae^{-x} = e^{-x}$$

$$-3Ae^{-x} + 2Ae^{-x} = e^{-x}$$

$$-Ae^{-x} = e^{-x}$$

$$A = -1$$

Hence the particular part of the solution is

$$y_P = -e^{-x}$$

The complete solution is given by

$$y = y_H + y_P$$

$$= Ce^{-0.666667x} - e^{-x}$$

The constant  $C$  can be obtained by using the initial condition  $y(0) = 5$

$$y(0) = Ce^{-0.666667 \times 0} - e^{-0} = 5$$

$$C - 1 = 5$$

$$C = 6$$

The complete solution is

$$y = 6e^{-0.666667x} - e^{-x}$$

## Example 2

Solve

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$$2\frac{dy}{dx} + 3y = e^{-1.5x}, \quad y(0) = 5$$

### Solution

The homogeneous solution for the above equation is given by

$$(2D + 3)y = 0$$

The characteristic equation for the above equation is given by

$$2r + 3 = 0$$

The solution to the equation is

$$r = -1.5$$

$$y_H = Ce^{-1.5x}$$

Based on the forcing function of the ordinary differential equations, the particular part of the solution is of the form  $Ae^{-1.5x}$ , but since that is part of the form of the homogeneous part of the solution, we need to choose the next independent solution, that is,

$$y_P = Axe^{-1.5x}$$

To find  $A$ , we substitute this solution in the ordinary differential equation as

$$2\frac{d(Axe^{-1.5x})}{dx} + 3Axe^{-1.5x} = e^{-1.5x}$$

$$2Ae^{-1.5x} - 3Axe^{-1.5x} + 3Axe^{-1.5x} = e^{-1.5x}$$

$$2Ae^{-1.5x} = e^{-1.5x}$$

$$A = 0.5$$

Hence the particular part of the solution is

$$y_P = 0.5xe^{-1.5x}$$

The complete solution is given by

$$y = y_H + y_P$$

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$$= Ce^{-1.5x} + 0.5xe^{-1.5x}$$

The constant  $C$  is obtained by using the initial condition  $y(0) = 5$ .

$$y(0) = Ce^{-1.5(0)} + 0.5(0)e^{-1.5(0)} = 5$$

$$C + 0 = 5$$

$$C = 5$$

The complete solution is

$$y = 5e^{-1.5x} + 0.5xe^{-1.5x}$$

### Sample Case 2

When

$$X = \sin(ax) \text{ or } \cos(ax),$$

the particular part of the solution is of the form

$$A \sin(ax) + B \cos(ax).$$

We can get  $A$  and  $B$  by substituting  $y = A \sin(ax) + B \cos(ax)$  in the left hand side of the differential equation and equating coefficients.

### **Example 3**

Solve

$$2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 3.125y = \sin x, \quad y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

### **Solution**

The homogeneous equation is given by

$$(2D^2 + 3D + 3.125)y = 0$$

The characteristic equation is

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$$2r^2 + 3r + 3.125 = 0$$

The roots of the characteristic equation are

$$\begin{aligned} r &= \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times 3.125}}{2 \times 2} \\ &= \frac{-3 \pm \sqrt{9 - 25}}{4} \\ &= \frac{-3 \pm \sqrt{-16}}{4} \\ &= \frac{-3 \pm 4i}{4} \\ &= -0.75 \pm i \end{aligned}$$

Therefore the homogeneous part of the solution is given by

$$y_H = e^{-0.75x} (K_1 \cos x + K_2 \sin x)$$

The particular part of the solution is of the form

$$\begin{aligned} y_p &= A \sin x + B \cos x \\ 2 \frac{d^2}{dx^2} (A \sin x + B \cos x) + 3 \frac{d}{dx} (A \sin x + B \cos x) + 3.125(A \sin x + B \cos x) &= \sin x \\ 2 \frac{d}{dx} (A \cos x - B \sin x) + 3(A \cos x - B \sin x) + 3.125(A \sin x + B \cos x) &= \sin x \\ 2(-A \sin x - B \cos x) + 3(A \cos x - B \sin x) + 3.125(A \sin x + B \cos x) &= \sin x \\ (1.125A - 3B) \sin x + (1.125B + 3A) \cos x &= \sin x \end{aligned}$$

Equating coefficients of  $\sin x$  and  $\cos x$  on both sides, we get

$$1.125A - 3B = 1$$

$$1.125B + 3A = 0$$

Solving the above two simultaneous linear equations we get

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$$A = 0.109589$$

$$B = -0.292237$$

Hence

$$y_p = 0.109589 \sin x - 0.292237 \cos x$$

The complete solution is given by

$$y = e^{-0.75x} (K_1 \cos x + K_2 \sin x) + (0.109589 \sin x - 0.292237 \cos x)$$

To find  $K_1$  and  $K_2$  we use the initial conditions

$$y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

From  $y(0) = 5$  we get

$$5 = e^{-0.75(0)} (K_1 \cos(0) + K_2 \sin(0)) + (0.109589 \sin(0) - 0.292237 \cos(0))$$

$$5 = K_1 - 0.292237$$

$$K_1 = 5.292237$$

$$\begin{aligned} \frac{dy}{dx} &= -0.75e^{-0.75x} (K_1 \cos x + K_2 \sin x) + e^{-0.75x} (-K_1 \sin x + K_2 \cos x) \\ &\quad + 0.109589 \cos x + 0.292237 \sin x \end{aligned}$$

From

$$\frac{dy}{dx}(x=0) = 3,$$

we get

$$\begin{aligned} 3 &= -0.75e^{-0.75(0)} (K_1 \cos(0) + K_2 \sin(0)) + e^{-0.75(0)} (-K_1 \sin(0) + K_2 \cos(0)) \\ &\quad + 0.109589 \cos(0) + 0.292237 \sin(0) \end{aligned}$$

$$3 = -0.75K_1 + K_2 + 0.109589$$

$$3 = -0.75(5.292237) + K_2 + 0.109589$$

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$$K_2 = 6.859588$$

The complete solution is

$$y = e^{-0.75x}(5.292237 \cos x + 6.859588 \sin x) + 0.109589 \sin x - 0.292237 \cos x$$

#### Example 4

Solve

$$2 \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 3.125y = \cos(x), \quad y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

#### Solution

The homogeneous part of the equations is given by

$$(2D^2 + 6D + 3.125)y = 0$$

The characteristic equation is given by

$$2r^2 + 6r + 3.125 = 0$$

$$r = \frac{-6 \pm \sqrt{6^2 - 4(2)(3.125)}}{2(2)}$$

$$= \frac{-6 \pm \sqrt{36 - 25}}{4}$$

$$= \frac{-6 \pm \sqrt{11}}{4}$$

$$= -1.5 \pm 0.829156$$

$$= -0.670844, -2.329156$$

Therefore, the homogeneous solution  $y_H$  is given by

$$y_H = K_1 e^{-0.670845x} + K_2 e^{-2.329156x}$$

The particular part of the solution is of the form

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$$y_p = A \sin x + B \cos x$$

Substituting the particular part of the solution in the differential equation,

$$2 \frac{d^2}{dx^2} (A \sin x + B \cos x) + 6 \frac{d}{dx} (A \sin x + B \cos x) + 3.125(A \sin x + B \cos x) = \cos x$$

$$2 \frac{d}{dx} (A \cos x - B \sin x) + 6(A \cos x - B \sin x) + 3.125(A \sin x + B \cos x) = \cos x$$

$$2(-A \sin x - B \cos x) + 6(A \cos x - B \sin x) + 3.125(A \sin x + B \cos x) = \cos x$$
$$(1.125A - 6B) \sin x + (1.125B + 6A) \cos x = \cos x$$

Equating coefficients of  $\cos x$  and  $\sin x$  we get

$$1.125B + 6A = 1$$

$$1.125A - 6B = 0$$

The solution to the above two simultaneous linear equations are

$$A = 0.161006$$

$$B = 0.0301887$$

Hence the particular part of the solution is

$$y_p = 0.161006 \sin x + 0.0301887 \cos x$$

Therefore the complete solution is

$$y = y_H + y_P$$

$$y = (K_1 e^{-0.670845x} + K_2 e^{-2.329156x}) + 0.161006 \sin x + 0.0301887 \cos x$$

Constants  $K_1$  and  $K_2$  can be determined using initial conditions. From  $y(0) = 5$ ,

$$y(0) = K_1 + K_2 + 0.0301887 = 5$$

$$K_1 + K_2 = 5 - 0.0301887 = 4.969811$$

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Now

$$\frac{dy}{dx} = -0.670845K_1e^{-(0.670845)x} - 2.329156K_2e^{-(2.329156)x} + 0.161006 \cos x - 0.0301887 \sin x$$

From  $\frac{dy}{dx}(x=0) = 3$

$$-0.670845K_1 - 2.329156K_2 + 0.161006 = 3$$

$$0.670845K_1 + 2.329156K_2 = -3 + 0.161006$$

$$0.670845K_1 + 2.329156K_2 = -2.838994$$

We have two linear equations with two unknowns

$$K_1 + K_2 = 4.969811$$

$$0.670845K_1 + 2.329156K_2 = -2.838994$$

Solving the above two simultaneous linear equations, we get

$$K_1 = 8.692253$$

$$K_2 = -3.722442$$

The complete solution is

$$y = (8.692253e^{-0.670845x} - 3.722442e^{-2.329156x}) + 0.161006 \sin x + 0.0301887 \cos x.$$

### Sample Case 3

When

$$X = e^{ax} \sin bx \text{ or } e^{ax} \cos bx ,$$

the particular part of the solution is of the form

$$e^{ax} (A \sin bx + B \cos bx) ,$$

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we can get  $A$  and  $B$  by substituting

$$y = e^{ax} (A \sin bx + B \cos bx)$$

in the left hand side of differential equation and equating coefficients.

### Example 5

Solve

$$2 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 3.125y = e^{-x} \sin x, y(0) = 5, \frac{dy}{dx}(x=0) = 3$$

### Solution

The homogeneous equation is given by

$$(2D^2 + 5D + 3.125)y = 0$$

The characteristic equation is given by

$$2r^2 + 5r + 3.125 = 0$$

$$r = \frac{-5 \pm \sqrt{5^2 - 4(2)(3.125)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{25 - 25}}{4}$$

$$= \frac{-5 \pm 0}{4}$$

$$= -1.25, -1.25$$

Since roots are repeated, the homogeneous solution  $y_H$  is given by

$$y_H = (K_1 + K_2 x)e^{(-1.25)x}$$

The particular part of the solution is of the form

$$y_P = e^{-x} (A \sin x + B \cos x)$$

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Substituting the particular part of the solution in the ordinary differential equation

$$\begin{aligned}
 & 2 \frac{d^2}{dx^2} \{e^{-x}(A \sin x + B \cos x)\} + 5 \frac{d}{dx} \{e^{-x}(A \sin x + B \cos x)\} \\
 & \quad + 3.125 \{e^{-x}(A \sin x + B \cos x)\} = e^{-x} \sin x \\
 & 2 \frac{d}{dx} \{-e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)\} \\
 & \quad + 5 \{-e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)\} + 3.125 e^{-x}(A \sin x + B \cos x) = e^{-x} \sin x \\
 & 2\{e^{-x}(A \sin x + B \cos x) - e^{-x}(A \cos x - B \sin x) - e^{-x}(A \cos x - B \sin x) - e^{-x}(A \sin x + B \cos x)\} \\
 & \quad + 5\{-e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x)\} + 3.125 e^{-x}(A \sin x + B \cos x) = e^{-x} \sin x \\
 & -1.875 e^{-x}(A \sin x + B \cos x) + e^{-x}(A \cos x - B \sin x) = e^{-x} \sin x \\
 & -1.875(A \sin x + B \cos x) + (A \cos x - B \sin x) = \sin x \\
 & -(1.875A + B) \sin x + (A - 1.875B) \cos x = \sin x
 \end{aligned}$$

Equating coefficients of  $\cos x$  and  $\sin x$  on both sides we get

$$A - 1.875B = 0$$

$$1.875A + B = -1$$

Solving the above two simultaneous linear equations we get

$$A = -0.415224 \text{ and}$$

$$B = -0.221453$$

Hence,

$$y_p = -e^{-x}(0.415224 \sin x + 0.221453 \cos x)$$

Therefore complete solution is given by

$$y = y_H + y_p$$

$$y = (K_1 + xK_2)e^{-1.25x} - e^{-x}(0.415224 \sin x + 0.221453 \cos x)$$

Constants  $K_1$  and  $K_2$  can be determined using initial conditions,

From  $y(0) = 5$ , we get



$$K_1 - 0.221453 = 5$$

$$K_1 = 5.221453$$

Now

$$\frac{dy}{dx} = -1.25K_1e^{-1.25x} - 1.25K_2xe^{-1.25x} + K_2e^{-1.25x} - e^{-x}(0.415224 \cos x - 0.221453 \sin x) + e^{-x}(0.415224 \sin x + 0.221453 \cos x)$$

From  $\frac{dy}{dx}(0) = 3$ , we get

$$\begin{aligned} & -1.25K_1e^{-1.25(0)} - 1.25K_2(0)e^{-1.25(0)} + K_2e^{-1.25(0)} \\ & - e^0(0.415224 \cos(0) - 0.221453 \sin(0)) + e^0(0.415224 \sin(0) + 0.221453 \cos(0)) = 3 \\ & -1.25K_1 + K_2 + 0.221453 - 0.415224 = 3 \end{aligned}$$

$$-1.25K_1 + K_2 = 3.193771$$

$$-1.25(5.221453) + K_2 = 3.193771$$

$$K_2 = 9.720582$$

Substituting

$$K_1 = 5.221453 \text{ and}$$

$$K_2 = 9.720582$$

in the solution, we get

$$y = (5.221453 + 9.720582x)e^{-1.25x} - e^{-x}(0.415224 \sin x + 0.221453 \cos x)$$

The forms of the particular part of the solution for different right hand sides of ordinary differential equations are given below

$X$	$y_p(x)$



$a_0 + a_1x + a_2x^2$	$b_0 + b_1x + b_2x^2$
$e^{ax}$	$Ae^{ax}$
$\sin(bx)$	$A \sin(bx) + B \cos(bx)$
$e^{ax} \sin(bx)$	$e^{ax} (A \sin(bx) + B \cos(bx))$
$\cos(bx)$	$A \sin(bx) + B \cos(bx)$
$e^{ax} \cos(bx)$	$e^{ax} (A \sin(bx) + B \cos(bx))$

## Laplace Transforms

If  $y = f(x)$  is defined at all positive values of  $x$ , the Laplace transform denoted by  $Y(s)$  is given by

$$Y(s) = L\{f(x)\} = \int_0^{\infty} e^{-sx} f(x) dx \quad (35)$$

where  $s$  is a parameter, which can be a real or complex number. We can get back  $f(x)$  by taking the inverse Laplace transform of  $Y(s)$ .

$$L^{-1}\{Y(s)\} = f(x) \quad (36)$$

Laplace transforms are very useful in solving differential equations. They give the solution directly without the necessity of evaluating arbitrary constants separately.

The following are Laplace transforms of some elementary functions

$$L(1) = \frac{1}{s}$$

$$L(x^n) = \frac{n!}{s^{n+1}}, \text{ where } n = 0, 1, 2, 3, \dots$$

$$L(e^{ax}) = \frac{1}{s - a}$$

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$$L(\sin ax) = \frac{a}{s^2 + a^2}$$

$$L(\cos ax) = \frac{s}{s^2 + a^2}$$

$$L(\sinh ax) = \frac{a}{s^2 - a^2}$$

$$L(\cosh ax) = \frac{s}{s^2 - a^2} \quad (37)$$

The following are the inverse Laplace transforms of some common functions

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{ax}$$

$$L^{-1}\left(\frac{1}{s^n}\right) = \frac{x^{n-1}}{(n-1)!}, \text{ where } n = 1, 2, 3, \dots$$

$$L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{ax} x^{n-1}}{(n-1)!}$$

$$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin ax$$

$$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos ax$$

$$L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a} \sinh ax$$

$$L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at$$

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$$L^{-1}\left(\frac{1}{(s-a)^2 + b^2}\right) = \frac{1}{b} e^{ax} \sin bx$$

$$L^{-1}\left(\frac{s-a}{(s-a)^2 + b^2}\right) = e^{ax} \cos bx$$

$$L^{-1}\left(\frac{s}{(s^2 + a^2)^2}\right) = \frac{1}{2a} x \sin ax$$

(38)

## Properties of Laplace transforms

### Linear property

If  $a, b, c$  are constants and  $f(x), g(x)$ , and  $h(x)$  are functions of  $x$  then

$$L[af(x) + bg(x) + ch(x)] = aL(f(x)) + bL(g(x)) + cL(h(x)) \quad (39)$$

### Shifting property

If

$$L\{f(x)\} = Y(s) \quad (40)$$

then

$$L\{e^{at} f(x)\} = Y(s-a) \quad (41)$$

Using shifting property we get

$$L(e^{ax} x^n) = \frac{n!}{(s-a)^{n+1}}, \quad n \geq 0$$

$$L(e^{ax} \sin bx) = \frac{b}{(s-a)^2 + b^2}$$

$$L(e^{ax} \cos bx) = \frac{s-a}{(s-a)^2 + b^2}$$

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$$L(e^{ax} \sinh bx) = \frac{b}{(s-a)^2 - b^2}$$

$$L(e^{ax} \cosh bx) = \frac{s-a}{(s-a)^2 - b^2} \tag{42}$$

**Scaling property**

If

$$L\{f(x)\} = Y(s) \tag{43}$$

then

$$L\{f(ax)\} = \frac{1}{a} Y\left(\frac{s}{a}\right) \tag{44}$$

**Laplace transforms of derivatives**

If the first  $n$  derivatives of  $f(x)$  are continuous then

$$L\{f^n(x)\} = \int_0^\infty e^{-sx} f^n(x) dx \tag{45}$$

Using integration by parts we get

$$\begin{aligned} \int_0^\infty e^{-sx} f^n(x) dx &= \left[ e^{-sx} f^{n-1}(x) - (-s)e^{-sx} f^{n-2}(x) \right. \\ &\quad \left. + (-s)^2 e^{-sx} f^{n-3}(x) + \dots + (-1)^{n-1} (-s)^{n-1} e^{-sx} f(x) \right]_0^\infty \\ &\quad + (-1)^n (-s)^n \int_0^\infty e^{-sx} f(x) dx \\ &= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-sx} f(x) dx \\ &= s^n Y(s) - f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) \end{aligned} \tag{46}$$

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## Laplace transform technique to solve ordinary differential equations

The following are steps to solve ordinary differential equations using the Laplace transform method

- (A) Take the Laplace transform of both sides of ordinary differential equations.
  - (B) Express  $Y(s)$  as a function of  $s$ .
  - (C) Take the inverse Laplace transform on both sides to get the solution.
- Let us solve Examples 1 through 4 using the Laplace transform method.

### Example 6

Solve

$$3\frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = 5$$

### Solution

Taking the Laplace transform of both sides, we get

$$L\left(3\frac{dy}{dx} + 2y\right) = L(e^{-x})$$

$$3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

Using the initial condition,  $y(0) = 5$  we get

$$3[sY(s) - 5] + 2Y(s) = \frac{1}{s+1}$$

$$(3s + 2)Y(s) = \frac{1}{s+1} + 15$$

$$(3s + 2)Y(s) = \frac{15s + 16}{s+1}$$

$$Y(s) = \frac{15s + 16}{(s+1)(3s+2)}$$

Writing the expression for  $Y(s)$  in terms of partial fractions

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$$\frac{15s+16}{(s+1)(3s+2)} = \frac{A}{s+1} + \frac{B}{3s+2}$$

$$\frac{15s+16}{(s+1)(3s+2)} = \frac{3As+2A+Bs+B}{(s+1)(3s+2)}$$

$$15s+16 = 3As+2A+Bs+B$$

Equating coefficients of  $s^1$  and  $s^0$  gives

$$3A+B=15$$

$$2A+B=16$$

The solution to the above two simultaneous linear equations is

$$A = -1$$

$$B = 18$$

$$Y(s) = \frac{-1}{s+1} + \frac{18}{3s+2}$$

$$= \frac{-1}{s+1} + \frac{6}{s+0.666667}$$

Taking the inverse Laplace transform on both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left(\frac{-1}{s+1}\right) + L^{-1}\left(\frac{6}{s+0.666667}\right)$$

Since

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

The solution is given by

$$y(x) = -e^{-x} + 6e^{-0.666667x}$$

## Example 7

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Solve

$$2\frac{dy}{dx} + 3y = e^{-1.5x}, \quad y(0) = 5$$

### Solution

Taking the Laplace transform of both sides, we get

$$L\left(2\frac{dy}{dx} + 3y\right) = L(e^{-1.5x})$$

$$2[sY(s) - y(0)] + 3Y(s) = \frac{1}{s + 1.5}$$

Using the initial condition  $y(0) = 5$ , we get

$$2[sY(s) - 5] + 3Y(s) = \frac{1}{s + 1.5}$$

$$(2s + 3)Y(s) = \frac{1}{s + 1.5} + 10$$

$$(2s + 3)Y(s) = \frac{10s + 16}{s + 1.5}$$

$$\begin{aligned} Y(s) &= \frac{10s + 16}{(s + 1.5)(2s + 3)} \\ &= \frac{10s + 16}{2(s + 1.5)(s + 1.5)} \\ &= \frac{10s + 16}{2(s + 1.5)^2} \\ &= \frac{5s + 8}{(s + 1.5)^2} \end{aligned}$$

Writing the expression for  $Y(s)$  in terms of partial fractions

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$$\frac{5s+8}{(s+1.5)^2} = \frac{A}{s+1.5} + \frac{B}{(s+1.5)^2}$$

$$\frac{5s+8}{(s+1.5)^2} = \frac{As+1.5A+B}{(s+1.5)^2}$$

$$5s+8 = As+1.5A+B$$

Equating coefficients of  $s^1$  and  $s^0$  gives

$$A = 5$$

$$1.5A + B = 8$$

The solution to the above two simultaneous linear equations is

$$A = 5$$

$$B = 0.5$$

$$Y(s) = \frac{5}{s+1.5} + \frac{0.5}{(s+1.5)^2}$$

Taking the inverse Laplace transform on both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left(\frac{5}{s+1.5}\right) + L^{-1}\left(\frac{0.5}{(s+1.5)^2}\right)$$

Since

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-ax} \text{ and } L^{-1}\left(\frac{1}{(s+a)^2}\right) = xe^{-ax}$$

The solution is given by

$$y(x) = 5e^{-1.5x} + 0.5xe^{-1.5x}$$



### Example 8

Solve

$$2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 3.125y = \sin x, \quad y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

### Solution

Taking the Laplace transform of both sides

$$L\left(2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 3.125y\right) = L(\sin x)$$

and knowing

$$L\left(\frac{d^2 y}{dx^2}\right) = s^2 Y(s) - sy(0) - \frac{dy}{dx}(x=0)$$

$$L\left(\frac{dy}{dx}\right) = sY(s) - y(0)$$

$$L(\sin x) = \frac{1}{s^2 + 1}$$

we get

$$2\left[s^2 Y(s) - sy(0) - \frac{dy}{dx}(x=0)\right] + 3[sY(s) - y(0)] + 3.125Y(s) = \frac{1}{s^2 + 1}$$

$$2[s^2 Y(s) - 5s - 3] + 3[sY(s) - 5] + 3.125Y(s) = \frac{1}{s^2 + 1}$$

$$[s(2s + 3) + 3.125]Y(s) - 10s - 21 = \frac{1}{s^2 + 1}$$

$$[s(2s + 3) + 3.125]Y(s) = \frac{1}{s^2 + 1} + 10s + 21$$

$$[2s^2 + 3s + 3.125]Y(s) = \frac{22 + 10s^3 + 10s + 21s^2}{(s^2 + 1)}$$

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$$Y(s) = \frac{10s^3 + 21s^2 + 10s + 22}{(s^2 + 1)(2s^2 + 3s + 3.125)}$$

Writing the expression for  $Y(s)$  in terms of partial fractions

$$\begin{aligned} \frac{As + B}{(2s^2 + 3s + 3.125)} + \frac{Cs + D}{(s^2 + 1)} &= \frac{10s^3 + 21s^2 + 10s + 22}{(s^2 + 1)(2s^2 + 3s + 3.125)} \\ \frac{As^3 + As + Bs^2 + B + 2Cs^3 + 3Cs^2 + 3.125Cs + 2Ds^2 + 3Ds + 3.125D}{(2s^2 + 3s + 3.125)(s^2 + 1)} &= \frac{10s^3 + 21s^2 + 10s + 22}{(s^2 + 1)(2s^2 + 3s + 3.125)} \\ \frac{(A + 2C)s^3 + (B + 3C + 2D)s^2 + (A + 3.125C + 3D)s + (B + 3.125D)}{(s^2 + 1)(2s^2 + 3s + 3.125)} &= \frac{10s^3 + 21s^2 + 10s + 22}{(s^2 + 1)(2s^2 + 3s + 3.125)} \end{aligned}$$

Equating terms of  $s^3$ ,  $s^2$ ,  $s^1$  and  $s^0$  gives

$$A + 2C = 10$$

$$B + 3C + 2D = 21$$

$$A + 3.125C + 3D = 10$$

$$B + 3.125D = 22$$

The solution to the above four simultaneous linear equations is

$$A = 10.584474$$

$$B = 21.657534$$

$$C = -0.292237$$

$$D = 0.109589$$

Hence



$$Y(s) = \frac{10.584474s + 21.657534}{2s^2 + 3s + 3.125} + \frac{-0.292237s + 0.109589}{s^2 + 1}$$

$$(2s^2 + 3s + 3.125) = 2\{(s^2 + 1.5s + 0.5625) + 1\} = 2\{(s + 0.75)^2 + 1\}$$

$$\begin{aligned} Y(s) &= \frac{10.584474(s + 0.75) + 13.719179}{2\{(s + 0.75)^2 + 1\}} + \frac{-0.292237s + 0.109589}{s^2 + 1} \\ &= \frac{5.292237(s + 0.75)}{\{(s + 0.75)^2 + 1\}} + \frac{6.859589}{\{(s + 0.75)^2 + 1\}} - \frac{0.292237s}{(s^2 + 1)} + \frac{0.109589}{(s^2 + 1)} \end{aligned}$$

Taking the inverse Laplace transform of both sides

$$\begin{aligned} L^{-1}\{Y(s)\} &= L^{-1}\left(\frac{5.292237(s + 0.75)}{\{(s + 0.75)^2 + 1\}}\right) + L^{-1}\left(\frac{6.859589}{\{(s + 0.75)^2 + 1\}}\right) \\ &\quad - L^{-1}\left(\frac{0.292237s}{s^2 + 1}\right) + L^{-1}\left(\frac{0.109589}{s^2 + 1}\right) \end{aligned}$$

$$\begin{aligned} L^{-1}\{Y(s)\} &= 5.292237L^{-1}\left(\frac{s + 0.75}{\{(s + 0.75)^2 + 1\}}\right) + 6.859589L^{-1}\left(\frac{1}{\{(s + 0.75)^2 + 1\}}\right) \\ &\quad - 0.292237L^{-1}\left(\frac{s}{s^2 + 1}\right) + 0.109589L^{-1}\left(\frac{1}{s^2 + 1}\right) \end{aligned}$$

Since

$$L^{-1}\left(\frac{s + a}{(s + a)^2 + b^2}\right) = e^{-ax} \cos bx$$

$$L^{-1}\left(\frac{b}{(s + a)^2 + b^2}\right) = e^{-ax} \sin bx$$

$$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \sin ax$$

$$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos ax$$

The complete solution is





$$\begin{aligned}
 y(x) &= 5.292237e^{-0.75x} \cos x + 6.8595859e^{-0.75x} \sin x \\
 &\quad - 0.292237 \cos x + 0.109589 \sin x \\
 &= e^{-0.75x} (5.292237 \cos x + 6.859589 \sin x) - 0.292237 \cos x + 0.109589 \sin x
 \end{aligned}$$

### Example 9

Solve

$$2 \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 3.125y = \cos x, \quad y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

### Solution

Taking the Laplace transform of both sides

$$L\left(2 \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 3.125y\right) = L(\cos x)$$

and knowing

$$L\left(\frac{d^2 y}{dx^2}\right) = s^2 Y(s) - sy(0) - \frac{dy}{dx}(x=0)$$

$$L\left(\frac{dy}{dx}\right) = sY(s) - y(0)$$

$$L(\cos x) = \frac{s}{s^2 + 1}$$

we get

$$2\left[s^2 Y(s) - sy(0) - \frac{dy}{dx}(x=0)\right] + 6[sY(s) - y(0)] + 3.125Y(s) = \frac{s}{s^2 + 1}$$

$$2[s^2 Y(s) - 5s - 3] + 6[sY(s) - 5] + 3.125Y(s) = \frac{s}{s^2 + 1}$$

$$[s(2s + 6) + 3.125]Y(s) = \frac{s}{s^2 + 1} + 10s + 36$$

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$$[2s^2 + 6s + 3.125]Y(s) = \frac{36 + 10s^3 + 11s + 36s^2}{s^2 + 1}$$

$$Y(s) = \frac{10s^3 + 36s^2 + 11s + 36}{(s^2 + 1)(2s^2 + 6s + 3.125)}$$

Writing the expression for  $Y(s)$  in terms of partial fractions

$$\begin{aligned} \frac{As + B}{(2s^2 + 6s + 3.125)} + \frac{Cs + D}{(s^2 + 1)} &= \frac{10s^3 + 36s^2 + 11s + 36}{(s^2 + 1)(2s^2 + 6s + 3.125)} \\ \frac{As^3 + As + Bs^2 + B + 2Cs^3 + 6Cs^2 + 3.125Cs + 2Ds^2 + 6Ds + 3.125D}{(2s^2 + 6s + 3.125)(s^2 + 1)} &= \frac{10s^3 + 36s^2 + 11s + 36}{(s^2 + 1)(2s^2 + 6s + 3.125)} \\ \frac{(A + 2C)s^3 + (B + 6C + 2D)s^2 + (A + 3.125C + 6D)s + (B + 3.125D)}{(s^2 + 1)(2s^2 + 6s + 3.125)} &= \frac{10s^3 + 36s^2 + 11s + 36}{(s^2 + 1)(2s^2 + 6s + 3.125)} \end{aligned}$$

Equating terms of  $s^3$ ,  $s^2$ ,  $s^1$  and  $s^0$  gives

$$A + 2C = 10$$

$$B + 6C + 2D = 36$$

$$A + 3.125C + 6D = 11$$

$$B + 3.125D = 36$$

The solution to the above four simultaneous linear equations is

$$A = 9.939622$$

$$B = 35.496855$$

$$C = 0.0301886$$

$$D = 0.161006$$



Then

$$Y(s) = \frac{9.939622s + 35.496855}{2s^2 + 6s + 3.125} + \frac{0.0301886s + 0.161006}{s^2 + 1}$$

$$(2s^2 + 6s + 3.125) = 2\{(s^2 + 3s + 2.25) - 0.6875\} = 2\{(s + 1.5)^2 - 0.829156^2\}$$

$$Y(s) = \frac{9.939622(s + 1.5) + 20.587422}{2\{(s + 1.5)^2 - 0.829156^2\}} + \frac{0.0301886s + 0.161006}{s^2 + 1}$$

$$\begin{aligned} &= \frac{4.969811(s + 1.5)}{\{(s + 1.5)^2 - 0.829156^2\}} + \frac{10.293711}{\{(s + 1.5)^2 - 0.829156^2\}} \\ &\quad + \frac{0.0301886s}{s^2 + 1} + \frac{0.161006}{s^2 + 1} \end{aligned}$$

Taking the inverse Laplace transform on both sides

$$\begin{aligned} L^{-1}\{Y(s)\} &= L^{-1}\left(\frac{4.969811(s + 1.5)}{\{(s + 1.5)^2 - 0.829156^2\}}\right) + L^{-1}\left(\frac{10.293711}{\{(s + 1.5)^2 - 0.829156^2\}}\right) \\ &\quad + L^{-1}\left(\frac{0.0301886s}{s^2 + 1}\right) + L^{-1}\left(\frac{0.161006}{s^2 + 1}\right) \\ &= 4.969811L^{-1}\left(\frac{(s + 1.5)}{(s + 1.5)^2 - 0.829156^2}\right) + 10.293711L^{-1}\left(\frac{1}{(s + 1.5)^2 - 0.829156^2}\right) \\ &\quad + 0.0301886L^{-1}\left(\frac{s}{(s^2 + 1)}\right) + 0.161006L^{-1}\left(\frac{1}{(s^2 + 1)}\right) \end{aligned}$$

Since

$$L^{-1}\left(\frac{s + a}{(s + a)^2 - b^2}\right) = e^{-ax} \cosh bx$$

$$L^{-1}\left(\frac{1}{(s + a)^2 - b^2}\right) = \frac{1}{b} e^{-ax} \sinh bx$$

$$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin ax$$



$$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos ax$$

The complete solution is

$$\begin{aligned} y(x) &= 4.969811e^{-1.5x} \cosh(0.829156x) + \frac{10.293711}{0.829156} e^{-1.5x} \sinh(0.829156x) \\ &\quad + 0.0301886 \cos x + 0.161006 \sin x \\ &= e^{-1.5x} \left( 4.969811 \left( \frac{e^{0.829156x} + e^{-0.829156x}}{2} \right) + 12.414685 \left( \frac{e^{0.829156x} - e^{-0.829156x}}{2} \right) \right) \\ &\quad + 0.030188 \cos x + 0.161006 \sin x \\ &= e^{-1.5x} (8.692248e^{0.829156x} - 3.722437e^{-0.829156x}) + 0.0301886 \cos x \\ &\quad + 0.161006 \sin x \end{aligned}$$

### Example 10

Solve

$$2 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 3.125y = e^{-x} \sin x, \quad y(0) = 5, \quad \frac{dy}{dx}(x=0) = 3$$

### Solution

Taking the Laplace transform of both sides

$$L\left(2 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 3.125y\right) = L(e^{-x} \sin x)$$

knowing

$$L\left(\frac{d^2 y}{dx^2}\right) = s^2 Y(s) - sy(0) - \frac{dy}{dx}(x=0)$$

$$L\left(\frac{dy}{dx}\right) = sY(s) - y(0)$$



$$L(e^{-x} \sin x) = \frac{1}{(s+1)^2 + 1}$$

we get

$$2\left[s^2Y(s) - sy(0) - \frac{dy}{dx}(x=0)\right] + 5[sY(s) - y(0)] + 3.125Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$2[s^2Y(s) - 5s - 3] + 5[sY(s) - 5] + 3.125Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$[s(2s+5) + 3.125]Y(s) - 10s - 31 = \frac{1}{(s+1)^2 + 1}$$

$$[s(2s+5) + 3.125]Y(s) = \frac{1}{(s+1)^2 + 1} + 10s + 31$$

$$[2s^2 + 5s + 3.125]Y(s) = \frac{63 + 10s^3 + 82s + 51s^2}{s^2 + 2s + 2}$$

$$Y(s) = \frac{10s^3 + 51s^2 + 82s + 63}{(s^2 + 2s + 2)(2s^2 + 5s + 3.125)}$$

Writing the expression for  $Y(s)$  in terms of partial fractions

$$\frac{As + B}{2s^2 + 5s + 3.125} + \frac{Cs + D}{s^2 + 2s + 2} = \frac{10s^3 + 51s^2 + 82s + 63}{(s^2 + 2s + 2)(2s^2 + 5s + 3.125)}$$

$$\frac{2Cs^3 + 5Cs^2 + 3.125Cs + 2Ds^2 + 5Ds + 3.125D + As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B}{(2s^2 + 5s + 3.125)(s^2 + 2s + 2)}$$

$$= \frac{10s^3 + 51s^2 + 82s + 63}{(s^2 + 2s + 2)(2s^2 + 5s + 3.125)}$$

$$\frac{(2C + A)s^3 + (5C + 2D + 2A + B)s^2 + (3.125C + 5D + 2A + 2B)s + (3.125D + 2B)}{(s^2 + 2s + 2)(2s^2 + 5s + 3.125)}$$

$$= \frac{10s^3 + 51s^2 + 82s + 63}{(s^2 + 2s + 2)(2s^2 + 5s + 3.125)}$$

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Equating terms of  $s^3, s^2, s^1$  and  $s^0$  gives four simultaneous linear equations

$$2C + A = 10$$

$$5C + 2D + 2A + B = 51$$

$$3.125C + 5D + 2A + 2B = 82$$

$$3.125D + 2B = 63$$

The solution to the above four simultaneous linear equations is

$$A = 10.442906$$

$$B = 32.494809$$

$$C = -0.221453$$

$$D = -0.636678$$

Then

$$Y(s) = \frac{10.442906s + 32.494809}{2s^2 + 5s + 3.125} + \frac{-0.221453s - 0.636678}{s^2 + 2s + 2}$$

$$(2s^2 + 5s + 3.125) = 2\{(s^2 + 2.5s + 1.5625)\} = 2(s + 1.25)^2$$

$$Y(s) = \frac{10.442906(s + 1.25) + 19.441176}{2(s + 1.25)^2} + \frac{-0.221453(s + 1) - 0.415225}{(s + 1)^2 + 1}$$

$$= \frac{5.221453(s + 1.25)}{(s + 1.25)^2} + \frac{9.720588}{(s + 1.25)^2} - \frac{0.221453(s + 1)}{(s + 1)^2 + 1} - \frac{0.415225}{(s + 1)^2 + 1}$$

Taking the inverse Laplace transform on both sides

$$\begin{aligned} L^{-1}\{Y(s)\} &= L^{-1}\left(\frac{5.221453}{(s + 1.25)}\right) + L^{-1}\left(\frac{9.720588}{(s + 1.25)^2}\right) \\ &\quad - L^{-1}\left(\frac{0.221453(s + 1)}{(s + 1)^2 + 1}\right) - L^{-1}\left(\frac{0.415225}{(s + 1)^2 + 1}\right) \end{aligned}$$



$$\begin{aligned}
&= 5.221453L^{-1}\left(\frac{1}{(s+1.25)}\right) + 9.720588L^{-1}\left(\frac{1}{(s+1.25)^2}\right) \\
&\quad - 0.221453L^{-1}\left(\frac{(s+1)}{(s+1)^2+1}\right) - 0.415225L^{-1}\left(\frac{1}{(s+1)^2+1}\right)
\end{aligned}$$

Since

$$L^{-1}\left(\frac{s+a}{(s+a)^2+b^2}\right) = e^{-ax} \cos bx$$

$$L^{-1}\left(\frac{b}{(s+a)^2+b^2}\right) = e^{-ax} \sin bx$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-ax}$$

$$L^{-1}\left(\frac{1}{(s+a)^n}\right) = \frac{e^{-ax} x^{n-1}}{(n-1)!}$$

The complete solution is

$$\begin{aligned}
y(x) &= 5.221453e^{-1.25x} + 9.720588e^{-1.25x}x - 0.221453e^{-x} \cos x \\
&\quad - 0.415225e^{-x} \sin x \\
&= e^{-1.25x}(5.221453 + 9.720588x) + e^{-x}(-0.221453 \cos x - 0.415225 \sin x)
\end{aligned}$$

