***Civil Engineering Example of Newton's Divided Difference Interpolation***

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### Example 1

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 1.

**Table 1** Temperature vs. depth for a lake.

| Temperature, $T$ $(°\text{C})$ | Depth, $z$ $(\text{m})$ |
|---|---|
| 19.1 | 0 |
| 19.1 | –1 |
| 19 | –2 |
| 18.8 | –3 |
| 18.7 | –4 |
| 18.3 | –5 |
| 18.2 | –6 |
| 17.6 | –7 |
| 11.7 | –8 |
| 9.9 | –9 |
| 9.1 | –10 |

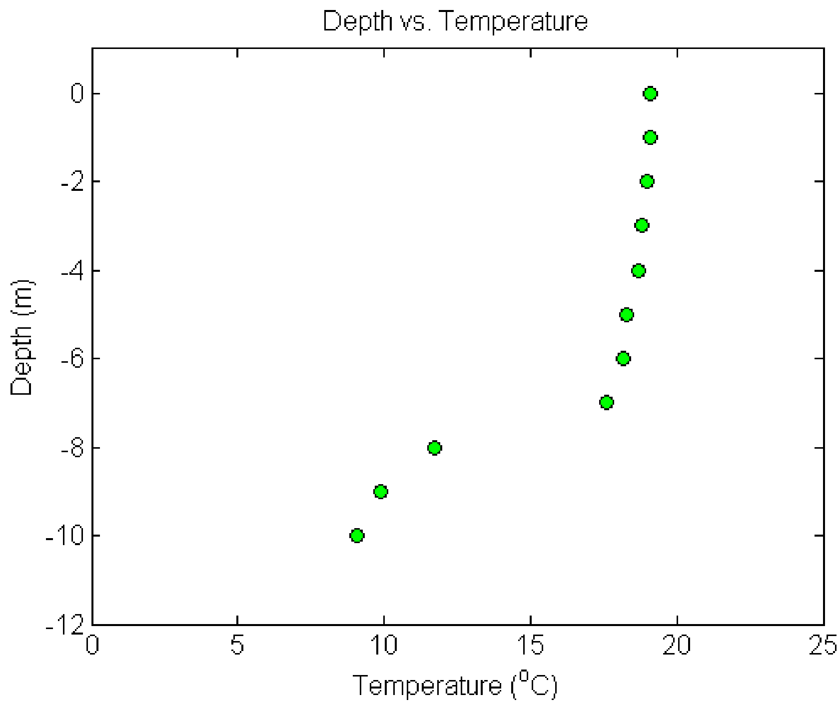


**Figure 1** Temperature vs. depth of a lake.

Using the given data, we see the largest change in temperature is between $z = -8\ \text{m}$ and $z = -7\ \text{m}$. Determine the value of the temperature at $z = -7.5\ \text{m}$ using Newton's divided difference method of interpolation and a first order polynomial.

**Solution**

For linear interpolation, the temperature is given by

$$T(z) = b_0 + b_1(z - z_0)$$

Since we want to find the temperature at $z = -7.5\ \text{m}$, and we are using a first order polynomial, we need to choose the two data points that are closest to $z = -7.5\ \text{m}$ that also bracket $z = -7.5\ \text{m}$ to evaluate it. The two points are $z_0 = -8$ and $z_1 = -7$.

Then

$$z_0 = -8,\ T(z_0) = 11.7$$

$$z_1 = -7,\ T(z_1) = 17.6$$

gives

$$b_0 = T(z_0)$$

$$= 11.7$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0}$$

$$= \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$

Hence

$$T(z) = b_0 + b_1(z - z_0)$$

$$= 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

At $z = -7.5$

$$T(-7.5) = 11.7 + 5.9(-7.5 + 8)$$

$$= 14.65\,°\text{C}$$

If we expand

$$T(z) = 11.7 + 5.9(z + 8), \qquad -8 \le z \le -7$$

we get

$$T(z) = 58.9 + 5.9z, \qquad -8 \le z \le -7$$

This is the same expression as obtained in the direct method.

**Example 2**

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 2.

**Table 2** Temperature vs. depth for a lake.

| Temperature, $T$ $(°C)$ | Depth, $z$ $(m)$ |
|---|---|
| 19.1 | 0 |
| 19.1 | –1 |
| 19 | –2 |
| 18.8 | –3 |
| 18.7 | –4 |
| 18.3 | –5 |
| 18.2 | –6 |
| 17.6 | –7 |
| 11.7 | –8 |
| 9.9 | –9 |
| 9.1 | –10 |

Using the given data, we see the largest change in temperature is between $z = -8 \text{ m}$ and $z = -7 \text{ m}$. Determine the value of the temperature at $z = -7.5 \text{ m}$ using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

**Solution**

For quadratic interpolation, the temperature is given by

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

Since we want to find the temperature at $z = -7.5$, and we are using a second order polynomial, we need to choose the three data points that are closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The three points are $z_0 = -9$, $z_1 = -8$ and $z_2 = -7$. (Choosing the three points as $z_0 = -8$, $z_1 = -7$ and $z_2 = -6$ is equally valid.)

Then

$$z_0 = -9,\ T(z_0) = 9.9$$

$$z_1 = -8,\ T(z_1) = 11.7$$

$$z_2 = -7,\ T(z_2) = 17.6$$

gives

$$b_0 = T(z_0)$$

$$= 9.9$$

$$b_1 = \frac{T(z_1) - T(z_0)}{z_1 - z_0}$$

$$= \frac{11.7 - 9.9}{-8 + 7}$$

$$= 1.8$$

$$b_2 = \frac{\dfrac{T(z_2) - T(z_1)}{z_2 - z_1} - \dfrac{T(z_1) - T(z_0)}{z_1 - z_0}}{z_2 - z_0}$$

$$= \frac{\dfrac{17.6 - 11.7}{-7 + 8} - \dfrac{11.7 - 9.9}{-8 + 9}}{-7 + 9}$$

$$= \frac{5.9 - 1.8}{2}$$

$$= 2.05$$

Hence

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1)$$

$$= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8), \ -9 \le z \le -7$$

At $z = -7.5,$

$$T(-7.5) = 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8)$$

$$= 14.138\,°\mathrm{C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{14.138 - 14.65}{14.138}\right| \times 100$$

$$= 3.6251\%$$

If we expand

$$T(z) = 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8), \quad -9 \le z \le -7$$

we get

$$T(z) = 173.7 + 36.65z + 2.05z^2, \qquad -9 \le z \le -7$$

This is the same expression obtained by the direct method.

**Example 3**

To maximize a catch of bass in a lake, it is suggested to throw the line to the depth of the thermocline. The characteristic feature of this area is the sudden change in temperature. We are given the temperature vs. depth data for a lake in Table 3.

**Table 3** Temperature vs. depth for a lake.

| Temperature, $T$ $(°\text{C})$ | Depth, $z$ $(\text{m})$ |
|---|---|
| 19.1 | 0 |
| 19.1 | –1 |
| 19 | –2 |
| 18.8 | –3 |
| 18.7 | –4 |
| 18.3 | –5 |
| 18.2 | –6 |
| 17.6 | –7 |
| 11.7 | –8 |
| 9.9 | –9 |
| 9.1 | –10 |

Using the given data, we see the largest change in temperature is between $z = -8\ \text{m}$ and $z = -7\ \text{m}$.

a) Determine the value of the temperature at $z = -7.5\ \text{m}$ using the direct method of interpolation and a third order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The position where the thermocline exists is given where $\frac{d^2T}{dz^2} = 0$. Using the expression from part (a), what is the value of the depth at which the thermocline exists?

**Solution**

a) For a third order polynomial, the temperature is given by

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

Since we want to find the temperature at $z = -7.5$, and we are using a third order polynomial, we need to choose the four data points that are closest to $z = -7.5$ that also bracket $z = -7.5$ to evaluate it. The four points are $z_0 = -9$, $z_1 = -8$, $z_2 = -7$ and $z_3 = -6$.

Then

$$z_0 = -9, \quad T(z_0) = 9.9$$

$$z_1 = -8, \quad T(z_1) = 11.7$$

$$z_2 = -7, \quad T(z_2) = 17.6$$

$$z_3 = -6, \quad T(z_3) = 18.2$$

gives

$$b_0 = T[z_0]$$

$$= T(z_0)$$

$$= 9.9$$

$$b_1 = T[z_1, z_0]$$

$$= \frac{T(z_1) - T(z_0)}{z_1 - z_0}$$

$$= \frac{11.7 - 9.9}{-8 + 9}$$

$$= 1.8$$

$$b_2 = T[z_2, z_1, z_0]$$

$$= \frac{T[z_2, z_1] - T[z_1, z_0]}{z_2 - z_0}$$

$$T[z_2, z_1] = \frac{T(z_2) - T(z_1)}{z_2 - z_1}$$

$$= \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$

$$T[z_1, z_0] = 1.8$$

$$b_2 = \frac{T[z_2, z_1] - T[z_1, z_0]}{z_2 - z_0}$$

$$= \frac{5.9 - 1.8}{-7 + 9}$$

$$= 2.05$$

$$b_3 = T[z_3, z_2, z_1, z_0]$$

$$= \frac{T[z_3, z_2, z_1] - T[z_2, z_1, z_0]}{z_3 - z_0}$$

$$T[z_3, z_2, z_1] = \frac{T[z_3, z_2] - T[z_2, z_1]}{z_3 - z_1}$$

$$T[z_3, z_2] = \frac{T(z_3) - T(z_2)}{z_3 - z_2}$$

$$= \frac{18.2 - 17.6}{-6 + 7}$$

$$= 0.6$$

$$T[z_2, z_1] = \frac{T(z_2) - T(z_1)}{z_2 - z_1}$$

$$= \frac{17.6 - 11.7}{-7 + 8}$$

$$= 5.9$$

$$T[z_3, z_2, z_1] = \frac{T[z_3, z_2] - T[z_2, z_1]}{z_3 - z_1}$$

$$= \frac{0.6 - 5.9}{-6 + 8}$$

$$= -2.65$$

$$T[z_2, z_1, z_0] = 2.05$$

$$b_3 = T[z_3, z_2, z_1, z_0]$$

$$= \frac{T[z_3, z_2, z_1] - T[z_2, z_1, z_0]}{z_3 - z_0}$$

$$= \frac{-2.65 - 2.05}{-6 + 9}$$

$$= -1.5667$$

Hence

$$T(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)(z - z_1) + b_3(z - z_0)(z - z_1)(z - z_2)$$

$$= 9.9 + 1.8(z + 9) + 2.05(z + 9)(z + 8) - 1.5667(z + 9)(z + 8)(z + 7), \; -9 \le z \le -6$$

At $z = -7.5$,

$$T(-7.5) = 9.9 + 1.8(-7.5 + 9) + 2.05(-7.5 + 9)(-7.5 + 8) - 1.5667(-7.5 + 9)(-7.5 + 8)(-7.5 + 7)$$

$$= 14.725\,^{\circ}\mathrm{C}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left| \frac{14.725 - 14.138}{14.725} \right| \times 100$$

$$= 3.9898\%$$

If we expand

$$T(z) = 9.9 + 1.8(z+9) + 2.05(z+9)(z+8) - 1.5667(z+9)(z+8)(z+7), \quad -9 \le z \le -6$$

we get

$$T(z) = -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

This is the same expression as obtained in the direct method.

b) To find the position of the thermocline, we must find the points of inflection of the third order polynomial, given by $\frac{d^2T}{dz^2} = 0$

$$T(z) = 9.9 + 1.8(z+9) + 2.05(z+9)(z+8) - 1.5667(z+9)(z+8)(z+7), \quad -9 \le z \le -6$$

$$= -615.9 - 262.58z - 35.55z^2 - 1.5667z^3, \quad -9 \le z \le -6$$

$$\frac{dT}{dz} = -262.58 - 71.1z - 4.7z^2, \quad -9 \le z \le -6$$

$$\frac{d^2T}{dz^2} = -71.1 - 9.4z, \quad -9 \le z \le -6$$

Simply setting this expression equal to zero, we get

$$0 = -71.1 - 9.4z$$

$$z = -7.5638 \text{ m}$$

This answer can be verified due to the fact that it falls within the specified range of the third order polynomial.