Industrial Engineering Example of the Direct Method of Interpolation

Autar Kaw

Example 1

The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

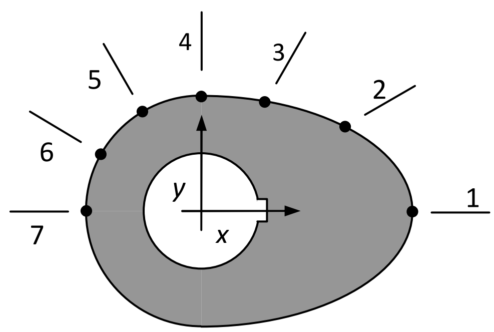

Figure 1 Schematic of cam profile.

Table 1 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using the direct method of interpolation and a first order polynomial?

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1 x$$

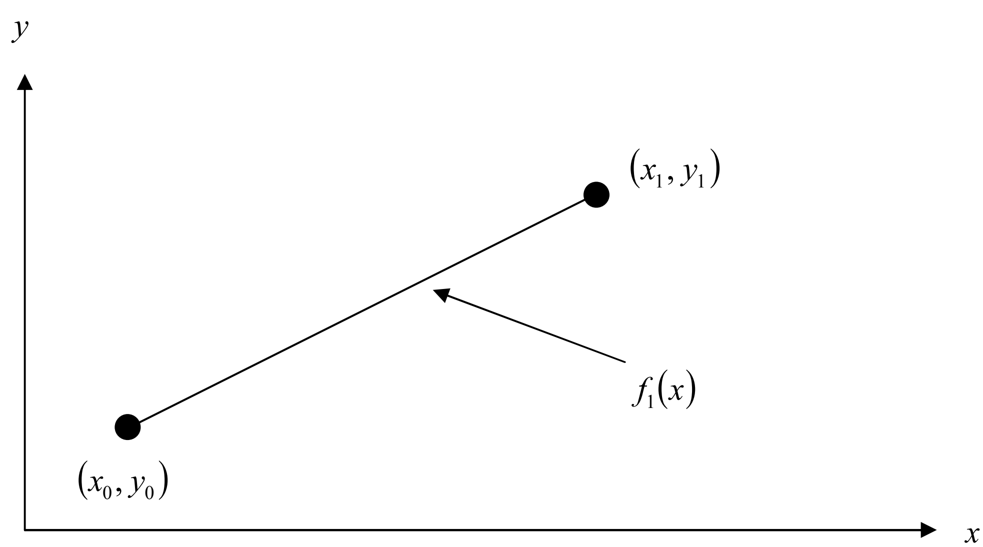

Figure 2 Linear interpolation.

Since we want to find the value of y at $x = 1.10$, and we are using a first order polynomial, using the two points $x_0 = 1.28$ and $x_1 = 0.66$, then

$$x_0 = 1.28, \ y(x_0) = 0.88$$

$$x_1 = 0.66, \ y(x_1) = 1.14$$

gives

$$y(1.28) = a_0 + a_1(1.28) = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) = 1.14$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 1.28 \\ 1 & 0.66 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.88 \\ 1.14 \end{bmatrix}$$

Solving the above two equations gives,

$$a_0 = 1.4168$$

$$a_1 = -0.41935$$

Hence

$$y(x) = a_0 + a_1 x$$

$$= 1.4168 - 0.41935x, \;\; 0.66 \le x \le 1.28$$

$$y(1.10) = 1.4168 - 0.41935(1.10)$$

$$= 0.95548 \text{ in.}$$

Example 2

The geometry of a cam is given in Figure 3. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

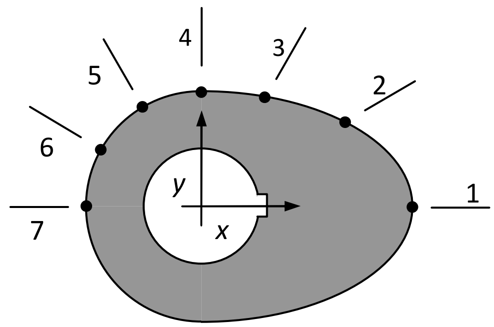

Figure 3 Schematic of cam profile.

Table 2 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a quadratic profile from $x = 2.20$ to $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using the direct method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1 x + a_2 x^2$$

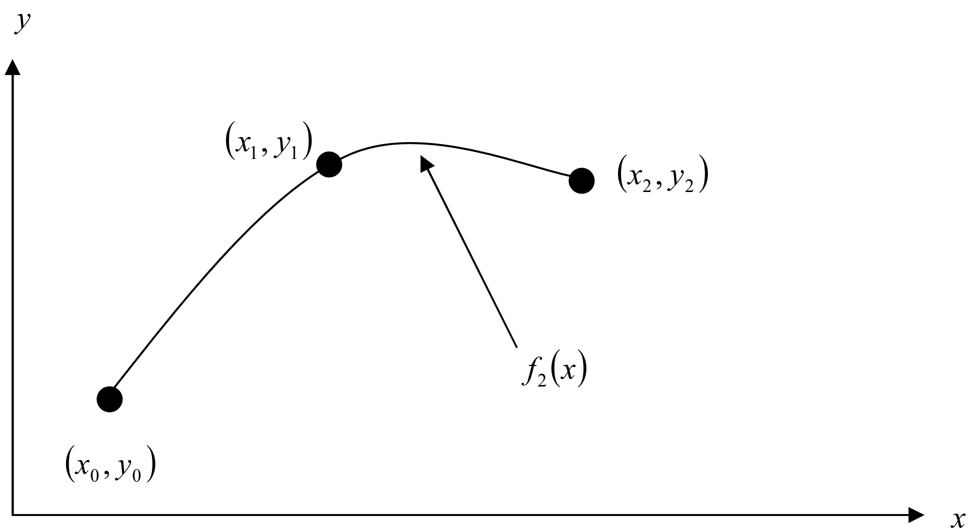

Figure 4 Quadratic interpolation.

Since we want to find the value of y at $x = 1.10$, and we are using a second order polynomial, using the three points $x_0 = 2.20$, $x_1 = 1.28$ and $x_2 = 0.66$, then

$$x_0 = 2.20, \quad y(x_0) = 0.00$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

gives

$$y(2.20) = a_0 + a_1(2.20) + a_2(2.20)^2 = 0.00$$

$$y(1.28) = a_0 + a_1(1.28) + a_2(1.28)^2 = 0.88$$

$$y(0.66) = a_0 + a_1(0.66) + a_2(0.66)^2 = 1.14$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.20 & 4.84 \\ 1 & 1.28 & 1.6384 \\ 1 & 0.66 & 0.4356 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.66 \\ 1.14 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 1.1221$$

$$a_1 = 0.25734$$

$$a_2 = -0.34881$$

Hence

$$y(x) = 1.1221 + 0.25734x - 0.34881x^2, \; 0.66 \le x \le 2.20$$

At $x = 1.10$,

$$y(1.10) = 1.1221 + 0.25734(1.10) - 0.34881(1.10)^2$$

$$= 0.98311 \text{ in}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100$$

$$= 2.8100\%$$

Example 3

The geometry of a cam is given in Figure 5. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.

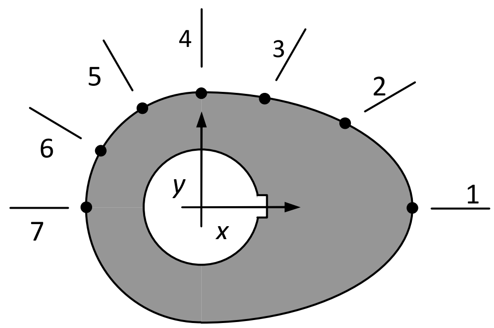

Figure 5 Schematic of cam profile.

Table 3 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

Find the cam profile using all seven points in Table 3 using the direct method of interpolation and a sixth order polynomial.

Solution

For the sixth order polynomial, we choose the value of y given by

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

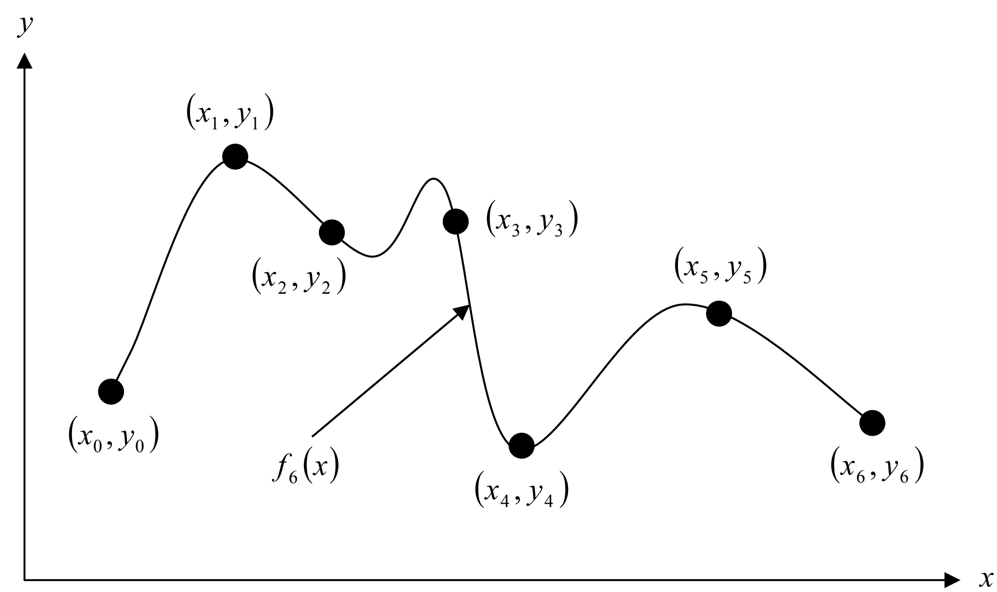

Figure 6 6th order polynomial interpolation.

Using the seven points,

$$x_0 = 2.20, \quad y(x_0) = 0$$

$$x_1 = 1.28, \quad y(x_1) = 0.88$$

$$x_2 = 0.66, \quad y(x_2) = 1.14$$

$$x_3 = 0.00, \quad y(x_3) = 1.20$$

$$x_4 = -0.60,\ \ y(x_4) = 1.04$$

$$x_5 = -1.04,\ \ y(x_5) = 0.60$$

$$x_6 = -1.20,\ \ y(x_6) = 0$$

gives

$$y(2.20) = 0.00 = a_0 + a_1(2.20) + a_2(2.20)^2 + a_3(2.20)^3 + a_4(2.20)^4 + a_5(2.20)^5 + a_6(2.20)^6$$

$$y(1.28) = 0.88 = a_0 + a_1(1.28) + a_2(1.28)^2 + a_3(1.28)^3 + a_4(1.28)^4 + a_5(1.28)^5 + a_6(1.28)^6$$

$$y(0.66) = 1.14 = a_0 + a_1(0.66) + a_2(0.66)^2 + a_3(0.66)^3 + a_4(0.66)^4 + a_5(0.66)^5 + a_6(0.66)^6$$

$$y(0.00) = 1.20 = a_0 + a_1(0.00) + a_2(0.00)^2 + a_3(0.00)^3 + a_4(0.00)^4 + a_5(0.00)^5 + a_6(0.00)^6$$

$$\begin{aligned} y(-0.60) = 1.04 = a_0 &+ a_1(-0.60) + a_2(-0.60)^2 + a_3(-0.60)^3 + a_4(-0.60)^4 \\ &+ a_5(-0.60)^5 + a_6(-0.60)^6 \end{aligned}$$

$$\begin{aligned} y(-1.04) = 0.60 = a_0 &+ a_1(-1.04) + a_2(-1.04)^2 + a_3(-1.04)^3 + a_4(-1.04)^4 \\ &+ a_5(-1.04)^5 + a_6(-1.04)^6 \end{aligned}$$

$$\begin{aligned} y(-1.20) = 0.00 = a_0 &+ a_1(-1.20) + a_2(-1.20)^2 + a_3(-1.20)^3 + a_4(-1.20)^4 \\ &+ a_5(-1.20)^5 + a_6(-1.20)^6 \end{aligned}$$

Writing the seven equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.20 & 2.20^2 & 2.20^3 & 2.20^4 & 2.20^5 & 2.20^6 \\ 1 & 1.28 & 1.28^2 & 1.28^3 & 1.28^4 & 1.28^5 & 1.28^6 \\ 1 & 0.66 & 0.66^2 & 0.66^3 & 0.66^4 & 0.66^5 & 0.66^6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.60 & 0.60^2 & -0.60^3 & 0.60^4 & -0.60^5 & 0.60^6 \\ 1 & -1.04 & 1.04^2 & -1.04^3 & 1.04^4 & -1.04^5 & 1.04^6 \\ 1 & -1.20 & 1.20^2 & -1.20^3 & 1.20^4 & -1.20^5 & 1.20^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.88 \\ 1.14 \\ 1.20 \\ 1.04 \\ 0.60 \\ 0.00 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 2.20 & 4.84 & 10.648 & 23.426 & 51.536 & 113.38 \\
1 & 1.28 & 1.6384 & 2.0972 & 2.6844 & 3.4360 & 4.3980 \\
1 & 0.66 & 0.4356 & 0.28750 & 0.18975 & 0.12523 & 0.082654 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.60 & 0.36 & -0.216 & 0.1296 & -0.07776 & 0.046656 \\
1 & -1.04 & 1.0816 & -1.1249 & 1.1699 & -1.2167 & 1.2653 \\
1 & -1.20 & 1.44 & -1.728 & 2.0736 & -2.4883 & 2.9860
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}
=
\begin{bmatrix} 0.00 \\ 0.88 \\ 1.14 \\ 1.20 \\ 1.04 \\ 0.60 \\ 0.00 \end{bmatrix}
$$

Solving the above seven equations gives

$$a_0 = 1.2$$

$$a_1 = 0.25112$$

$$a_2 = -0.27255$$

$$a_3 = -0.56765$$

$$a_4 = 0.072013$$

$$a_5 = 0.45241$$

$$a_6 = -0.17103$$

Hence

$$
\begin{aligned}
y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \\
&= 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \\
&\qquad + 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20
\end{aligned}
$$

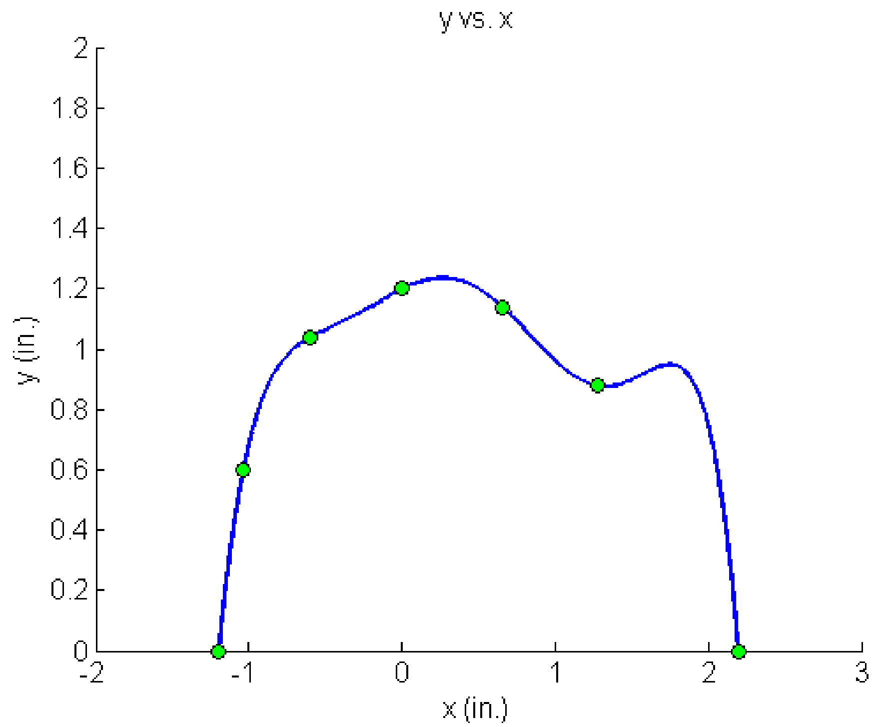

Figure 7 Plot of the cam profile as defined by a 6th order interpolating polynomial (using directed method of interpolation).