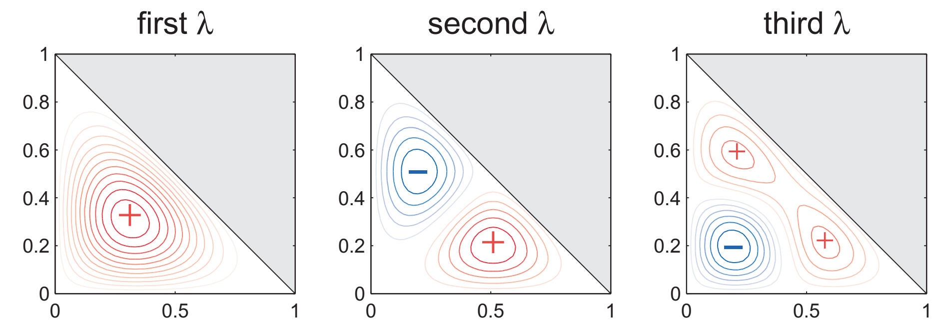

Figure 1: Contour plots of the eigenfunctions for the smallest three eigenvalues λ of the $-\nabla^2$ operator on a triangular half of a square (cut diagonally), with Dirichlet boundary conditions.

18.303 Problem Set 5

Due Wednesday, 13 October 2010.

Problem 1: (10+(5+5+5) points)

(a) The denominator of the Rayleigh quotient is $\langle u_a, u_a \rangle = \int_0^a (x/a)^2 dx + \int_a^1 (1-x)^2/(1-a)^2 dx = a/3 + (1-a)/3 = 1/3$. The derivative is $u_a'(x) = 1/a$ for $x \leq a$ and $1/(1-a)$ for $x > a$, so the numerator of the Rayleigh quotient is $\langle u_a', u_a' \rangle = \int_0^a (1/a)^2 dx + \int_a^1 1/(1-a)^2 dx = \frac{1}{a} + \frac{1}{1-a}$, and hence

$$R(u_a) = 3\left(\frac{1}{a} + \frac{1}{1-a}\right),$$

which is maximized when $\frac{dR}{da} = \frac{3}{(1-a)^2} - \frac{3}{a^2} = 0$, and hence at $\boxed{a = 1/2}$. The minimum is then $R(u_{1/2}) = 12$, which is a bit bigger than the smallest eigenvalue $\pi^2 \approx 9.8696$ (it is the right order of magnitude anyway, and as expected R cannot be *smaller* than the smallest λ).

(b) The contour plots are shown in figure 1. These are actually numerical calculations, but you should be able to guess something qualitatively similar. The reasoning goes as follows, similar to class. The eigenfunctions minimize $R(u) = \int |\nabla u|^2 / \int |u|^2$, subject to the boundary conditions and to orthogonality with the lower modes. Thus, they want to minimize the average $|\nabla u|$ relative to the overall amplitude $|u|$, i.e. they "want" to oscillate as slowly as possible.

The lowest λ solution should therefore just have a single peak in the center—it has to be zero at the boundaries, and must be nonzero somewhere, so the slowest it can oscillate is to go up to a peak in the center and then back down, as in figure 1(left).

The second u must be orthogonal to the first, so it must flip sign, i.e. have two peaks of opposite sign, but in which direction? This triangle is *not equilateral*[1]—it is longer in the $(-1, 1)$ direction (parallel to the diagonal of the square), so that is the direction in which a sign oscillation can occur most slowly. Hence, we should expect a single $+/-$ oscillation along this direction, as in figure 1(middle).

The third u must be orthogonal to the first two, and the slowest oscillation that will do this turns out to be in the $(1, 1)$ direction, as in figure 1(right). However, this one is a little tricky, because it is not completely obvious that the third u does not instead oscillate three times in the (-1,1) direction (i.e. $-/+/-$ peaks aong that direction), so I would accept that answer as well. [Exact calculations turn out to show that a 3-peak oscillation in the $(-1, 1)$

[1] In an equilateral triangle, the second two $\lambda's$ turn out to be equal, i.e. oscillating in either of the two directions gives the same Rayleigh queotient. This equality turns out to be a deeper consequence of symmetry, but that is outside the scope of 18.303.

direction gives the fourth λ, which is about 30% bigger than the third λ, corresponding to an average "wavelength" that is about 12% smaller.]

Problem 2: (10+10+5+5 points)

(a) The solutions in the three region I ($x < x'$), II ($x' \leq x \leq x' + \Delta x$), and III ($x > x' + \Delta x$) satisfy $\hat{A} g_{\text{I,III}} = 0$ and $\hat{A} g_{\text{II}} = 1/\Delta x$. Given the hint and the fact that they must vanish at $x = 0, L$, this means that they must be of the form:

$$g_{\text{I}} = \alpha(e^{\kappa x} - e^{-\kappa x})/2 = \alpha \sinh(\kappa x),$$

$$g_{\text{III}} = \beta \sinh(\kappa(x - L)),$$

$$g_{\text{II}} = \gamma e^{-\kappa(x-x')} + \xi e^{+\kappa(x-x')} + \frac{1}{\kappa^2 \Delta x}$$

for some constants $\alpha, \beta, \gamma, \xi$ to be determined (since these are free parameters, we can pull out convenient factors of 2, $e^{\kappa x'}$, etcetera if we want). Continuity of g and g' result in the following four equations:

$$\alpha \sinh(\kappa x') = \gamma + \xi + \frac{1}{\kappa^2 \Delta x}$$

$$\kappa \alpha \cosh(\kappa x') = \kappa(\xi - \gamma)$$

$$\beta \sinh(\kappa(x' + \Delta x - L)) = \gamma e^{-\kappa \Delta x} + \xi e^{+\kappa \Delta x} + \frac{1}{\kappa^2 \Delta x}$$

$$\kappa \beta \cosh(\kappa(x' + \Delta x - L)) = \kappa \left(\xi e^{+\kappa \Delta x} - \gamma e^{-\kappa \Delta x} \right).$$

Dividing the first equation by the second and the third by the fourth, and let $x'' = x' + \Delta x - L$, to yield two equations for γ and ξ:

$$\tanh(\kappa x') = \frac{\gamma + \xi + \frac{1}{\kappa^2 \Delta x}}{\xi - \gamma},$$

$$\tanh(\kappa x'') = \frac{\gamma e^{-\kappa \Delta x} + \xi e^{+\kappa \Delta x} + \frac{1}{\kappa^2 \Delta x}}{\xi e^{+\kappa \Delta x} - \gamma e^{-\kappa \Delta x}},$$

which (moving the denominators to the left-hand sides) becomes two linear equations in two unknowns. The first equation can be solved for γ to obtain:

$$\gamma = \frac{\xi[\tanh(\kappa x') - 1] - \frac{1}{\kappa^2 \Delta x}}{\tanh(\kappa x') + 1} = \xi T - \frac{p}{\kappa^2 \Delta x},$$

defining the quantities T and p for convenience. This can be substituted into the second equation to find ξ:

$$\xi = \frac{1}{\kappa^2 \Delta x} \frac{p e^{-\kappa \Delta x}[\tanh(\kappa x'') + 1] - 1}{e^{+\kappa \Delta x} + T e^{-\kappa \Delta x} - \tanh(\kappa x'') \left[e^{+\kappa \Delta x} - T e^{-\kappa \Delta x} \right]}.$$

Given γ and ξ, we then immediately have α and β:

$$\alpha = \frac{\xi - \gamma}{\cosh(\kappa x')}, \qquad \beta = \frac{\xi e^{+\kappa \Delta x} - \gamma e^{-\kappa \Delta x}}{\cosh(\kappa x'')}.$$

We can then plot this with the Matlab function:

```
function g = pset5g(kappa, L, x,xp,dx)

xpp = xp + dx - L;
T = (tanh(kappa*xp) - 1) / (tanh(kappa*xp) + 1);
p = 1 / (tanh(kappa*xp) + 1);
```

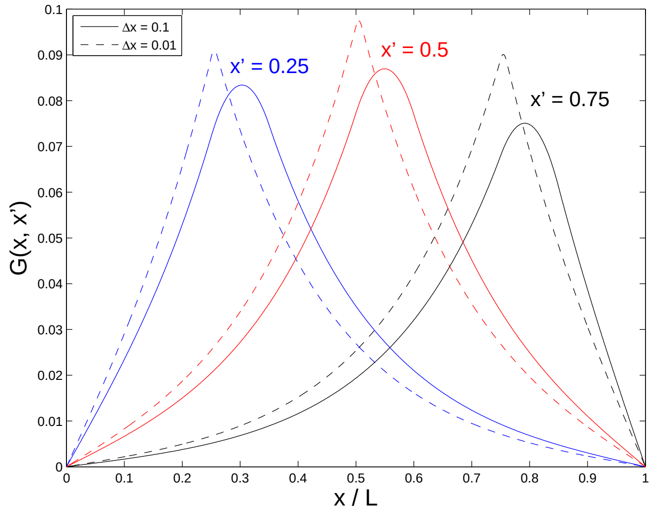

Figure 2: Green's function $G(x, x')$ for "approximate δ" consisting of width-Δx rectangular bump, plotted for $x' = 0.25$, 0.5, and 0.75, and for $\Delta x = 0.1$ and 0.01.

```
ep = exp(kappa*dx); em = exp(-kappa*dx);
xi = (p * em * (t + 1) - 1) / (kappa^2*dx) / ((ep+T*em) - t*(ep-T*em));
gam = xi * T - p / (kappa^2*dx);
alpha = (gam + xi) / sinh(kappa*xp);
beta = (gam*em + xi*ep) / sinh(kappa*xpp);
g = alpha*sinh(kappa*x) .* (x < xp) + beta*sinh(kappa*(x-L)) .* (x > xp+dx) ...
+ (gam*exp(-kappa*(x-xp))+xi*exp(kappa*(x-xp)) + 1/(kappa^2*dx)) ...
.* (x >= xp & x <= xp+dx);
```

which is shown in figure 2 for $\kappa = 5$. ($\kappa = 1$ is also acceptable, since that is what I asked for originally before I revised the problem via email. $\kappa = 1$ is similar but gives lines that are more straight; it is harder to see the exponentials.) Notice that the curves are clearly continuous with continuous slope—if we didn't get this, it would mean we made a mistake in solving our equations.

Even easier alternative: Since the equations are linear, it is perfectly acceptable to simply set them up as a 4×4 matrix in Matlab (or 6×6, including the boundary conditions at 0 and L) and ask Matlab to solve them for you.

(b) **Hard way:** In the limit $\Delta x \to 0$, then many things simplify: $e^{\pm\kappa\Delta x} \to 1$, $x'' \to x' - L$, and we obtain:

$$\xi \to \frac{1}{\kappa^2 \Delta x} \frac{\frac{\tanh(\kappa x'')+1}{\tanh(\kappa x')+1} - 1}{1 + T - \tanh(\kappa x'')[1-T]} = \frac{1}{\kappa^2 \Delta x} \frac{\tanh(\kappa x'') - \tanh(\kappa x')}{2\left[\tanh(\kappa x') - \tanh(\kappa x'')\right]} = -\frac{1}{2\kappa^2 \Delta x},$$

$$\gamma \to \frac{1}{\kappa^2 \Delta x} \frac{-\frac{1}{2}[\tanh(\kappa x') - 1] - 1}{\tanh(\kappa x') + 1} = -\frac{1}{2\kappa^2 \Delta x}.$$

Unfortunately, this is a little too simplified, because we need to compute $\xi - \gamma$ to get α and β, so we need to keep the next order in the Taylor expansion (or equivalently apply L'Hopital's rule to get α and β). To get the next-order term, we expand $e^{\pm\kappa\Delta x} \approx 1 \pm \kappa\Delta x$, to obtain:

$$\begin{aligned}
\xi &\approx -\frac{1}{\kappa^2 \Delta x} \frac{(1-\kappa\Delta x)[\tanh(\kappa x'') + 1] - [\tanh(\kappa x') + 1]}{(1+\kappa\Delta x)[\tanh(\kappa x') + 1]\,[1 - \tanh(\kappa x'')] + (1-\kappa\Delta x)[\tanh(\kappa x') - 1]\,[1 + \tanh(\kappa x'')]} \\
&\approx -\frac{1}{2\kappa^2 \Delta x} - \frac{1}{2\kappa}\left[\frac{\tanh(\kappa x'') + 1}{\tanh(\kappa x') - \tanh(\kappa x'')} - \frac{1 - \tanh(\kappa x')\tanh(\kappa x'')}{\tanh(\kappa x') - \tanh(\kappa x'')}\right] \\
&= -\frac{1}{2\kappa^2 \Delta x} - \frac{1}{2\kappa}\frac{\tanh(\kappa x') + 1}{\tanh(\kappa x') - \tanh(\kappa x'')}\tanh(\kappa x''),
\end{aligned}$$

in which case

$$\gamma \approx -\frac{1}{2\kappa^2 \Delta x} - \frac{1}{2\kappa}\frac{\tanh(\kappa x') - 1}{\tanh(\kappa x') - \tanh(\kappa x'')}\tanh(\kappa x''),$$

and thus we obtain

$$\begin{aligned}
\alpha &\to -\frac{1}{\kappa}\frac{1}{\cosh(\kappa x')\,[\tanh(\kappa x') - \tanh(\kappa x'')]}\tanh(\kappa x'') \\
&= \frac{\sinh(\kappa x'')/\kappa}{\sinh(\kappa x'')\cosh(\kappa x') - \cosh(\kappa x'')\sinh(\kappa x')}
\end{aligned}$$

and

$$\beta \to \frac{\sinh(\kappa x')/\kappa}{\sinh(\kappa x'')\cosh(\kappa x') - \cosh(\kappa x'')\sinh(\kappa x')}.$$

Easier way: It is *much* easier, however to solve for $G(x, x')$ directly, as in class. We write $\hat{A}G = \delta(x - x')$, and break it into two regions: I ($x < x'$) and II ($x > x''$) where $\hat{A}G = 0$. The solutions in these regions are, as above, of the form:

$$G_{\text{I}} = \alpha \sinh(\kappa x),$$

$$G_{\text{III}} = \beta \sinh(\kappa(x - L)).$$

We require G to be continuous at $x = x'$, but the slope of G' should drop discontinuously by 1 at x' to give us a $\delta(x - x')$ from $-G''$, leading to the equations:

$$\begin{aligned}
\alpha \sinh(\kappa x') &= \beta \sinh(\kappa x'') \\
\kappa\alpha \cosh(\kappa x') &= \kappa\beta \cosh(\kappa x'') + 1,
\end{aligned}$$

where $x'' = x' - L$. These two equations can easily be solved for α and β to yield:

$$\alpha = \frac{\sinh(\kappa x'')/\kappa}{\sinh(\kappa x'')\cosh(\kappa x') - \cosh(\kappa x'')\sinh(\kappa x')} = \frac{\sinh(\kappa[L - x'])}{\kappa \sinh(\kappa L)},$$

$$\beta = -\frac{\sinh(\kappa x')}{\kappa \sinh(\kappa L)},$$

using the identity $\sinh(A - B) = \sinh(A)\cosh(B) - \cosh(A)\sinh(B)$, as above. When plotted, this is visually almost indistinguishable from the $\Delta x = 0.01$ curves plotted in figure 2, so I won't replot it here to save space. [It is a good check to plot the curves on top of one another; you'll find that the $\Delta x = 0.01$ curves are only slightly shifted from $\Delta x \to 0$, by a percent or two.]

(c) From above, using the identity that $\sinh(-A) = \sinh(A)$, the Green's function is given by:

$$G(x, x') = \frac{1}{\kappa \sinh(\kappa L)} \begin{cases} \sinh(\kappa[L - x'])\sinh(\kappa x) & x < x' \\ \sinh(\kappa x')\sinh(\kappa[L - x]) & x > x' \end{cases}.$$

which by inspection is invariant under interchange of x and x' (which just swaps the two rows).

(d) $\hat{A}G(x,x') = \delta(x-x')$ by construction if we computed $G(x,x')$ the easier way, above, since $\hat{A}G = 0$ everywhere except that G' jumps by -1 at x', giving a $\delta(x-x')$ in $-G''$.

(e) *(This part was optional.)* Here,

$$
\begin{aligned}
u(x) &= \frac{1}{\kappa \sinh(\kappa L)} \left[\sinh(\kappa[L-x']) \int_0^{x'} \sinh(\kappa x) e^x dx + \sinh(\kappa x') \int_{x'}^{L} \sinh(\kappa[L-x]) e^x dx \right] \\
&= \frac{1}{2\kappa \sinh(\kappa L)} \left[\sinh(\kappa[L-x']) \left(\frac{e^{x(1+\kappa)}}{1+\kappa} - \frac{e^{x(1-\kappa)}}{1-\kappa} \right)\Bigg|_0^{x'} + \sinh(\kappa x') \left(e^{\kappa L} \frac{e^{x(1-\kappa)}}{1-\kappa} - e^{-\kappa L} \frac{e^{x(1+\kappa)}}{1+\kappa} \right)\Bigg|_{x'}^{L} \right] \\
&= \frac{1}{2\kappa \sinh(\kappa L)} \left[\sinh(\kappa[L-x']) \left(\frac{e^{x(1+\kappa)}}{1+\kappa} - \frac{e^{x(1-\kappa)}}{1-\kappa} \right)\Bigg|_0^{x'} + \sinh(\kappa x') \left(e^{\kappa L} \frac{e^{x(1-\kappa)}}{1-\kappa} - e^{-\kappa L} \frac{e^{x(1+\kappa)}}{1+\kappa} \right)\Bigg|_{x'}^{L} \right] \\
&= \frac{1}{2\kappa \sinh(\kappa L)} \left[\sinh(\kappa[L-x']) \left(\frac{e^{x'(1+\kappa)}}{1+\kappa} - \frac{e^{x'(1-\kappa)}}{1-\kappa} - \frac{2}{1-\kappa^2} \right) \right. \\
&\qquad \left. + \sinh(\kappa x') \left(\frac{2e^L}{1-\kappa^2} - e^{\kappa L} \frac{e^{x'(1-\kappa)}}{1-\kappa} + e^{-\kappa L} \frac{e^{x'(1+\kappa)}}{1+\kappa} \right) \right]
\end{aligned}
$$

which can probably simplify further, but I'll stop here.

Problem 3: (15 points)

Since this function has two discontinuities, at $x = \pm\Delta x$, and is otherwise constant, the derivative is simply two delta functions multiplied by the respective amplitudes of the discontinuities:

$$s'_{\Delta x} = \frac{1}{2\Delta x} \left[\delta(x-\Delta x) - \delta(x+\Delta x) \right],$$

which can also be seen by writing $s_{\Delta x}(x) = \frac{1}{2\Delta x} [S(x-\Delta x) - S(x+\Delta x)]$ in terms of the unit step function $S(x)$ from the notes and then applying the rule for differentiating S as a distribution. Hence,

$$s'_{\Delta x}\{\phi\} = \frac{1}{2\Delta x} \left[\phi(-\Delta x) - \phi(\Delta x) \right].$$

But this is exactly a center-difference approximation for $-\phi'(0)$. Therefore,

$$\lim_{\Delta x \to 0} s'_{\Delta x}\{\phi\} = -\lim_{\Delta x \to 0} \frac{\phi(\Delta x) - \phi(-\Delta x)}{2\Delta x} = -\phi'(0) = \delta'\{\phi\}$$

as desired.

18.303 Linear Partial Differential Equations: Analysis and Numerics
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