18.303 Problem Set 2 Solutions

Problem 1: (10+10 points)

We are using a difference approximation of the form:

$$u'(x) \approx \frac{-u(x+2\Delta x) + c \cdot u(x+\Delta x) - c \cdot u(x-\Delta x) + u(x-2\Delta x)}{d \cdot \Delta x}.$$

(a) First, we Taylor expand:

$$u(x+\Delta x) = \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!} \Delta x^n.$$

The numerator of the difference formula flips sign if $\Delta x \to -\Delta x$, which means that when you plug in the Taylor series all of the even powers of Δx must cancel! To get 4th-order accuracy, the Δx^3 term in the numerator (which would give an error $\sim \Delta x^2$) must cancel as well, and this determines our choice of c: the Δx^3 term in the numerator is

$$\frac{u'''(x)}{3!} \Delta x^3 \left[-2^3 + c + c - 2^3\right],$$

and hence we must have $\boxed{c = 2^3 = 8}$. The remaining terms in the numerator are the Δx term and the Δx^5 term:

$$u'(x)\Delta x\left[-2 + c + c - 2\right] + \frac{u^{(5)}(x)}{5!}\Delta x^5 \left[-2^5 + c + c - 2^5\right] = 12u'(x)\Delta x - \frac{2}{5}u^{(5)}(x)\Delta x^5 + \cdots.$$

Clearly, to get the correct $u'(x)$ as $\Delta x \to 0$, we must have $\boxed{d = 12}$. Hence, the error is approximately $-\frac{1}{30}u^{(5)}(x)\Delta x^4$, which is $\sim \Delta x^4$ as desired.

(b) The Matlab code is the same as in the handout, except now we compute our difference approximation by the command: d = `(-sin(x+2*dx) + 8*sin(x+dx) - 8*sin(x+dx) + sin(x-2*dx)) ./ (12 * dx)`; the result is plotted in Fig. 1. Note that the error falls as a straight line (a power law), until it reaches $\sim 10^{-15}$, when it starts becoming dominated by roundoff errors (and actually gets worse). To verify the order of accuracy, it would be sufficient to check the slope of the straight-line region, but it is more fun to plot the actual predicted error from the previous part, where $\frac{d^5}{dx^5}\sin(x) = -\cos(x)$. Clearly the predicted error is almost exactly right (until roundoff errors take over).

Problem 2: (10+10+(5+10) points)

(a) As in class, we integrate by parts:

$$\langle u, v'' \rangle = \int_0^L uv'' = uv'|_0^L - \int_0^L u'v' = -\left. u'v\right|_0^L + \int_0^L u''v = \langle u'', v \rangle,$$

where the boundary terms vanish because u' and v' are zero. As in class, integrating by parts only a single time gives

$$\langle u, -u'' \rangle = \langle u', u' \rangle = \|u'\|^2 \geq 0,$$

so this is positive semidefinite. It is not positive defintie because it can $= 0$ for $u(x) =$ constant $\neq 0$, which satisfies the boundary conditions.

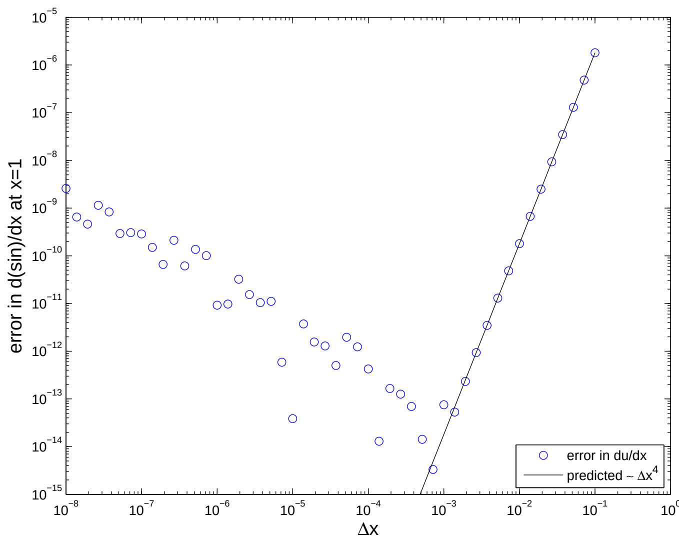

Figure 1: Actual vs. predicted error for problem 1(b), using fourth-order difference approximation for $u'(x)$ with $u(x) = \sin(x)$, at $x = 1$.

(b) As in class, we integrate by parts twice, but this time we also have to move the $\frac{1}{w(x)}$ term over (since this is just a real number, it is symmetric). (The w term doesn't change the fact that the boundary terms vanish, since u and v are still zero there regardless of what you multiply them by.)

$$\langle u, -\frac{1}{w}v''\rangle = \langle\frac{1}{w}u, -v''\rangle = \langle\frac{d}{dx}\frac{u}{w}, v'\rangle = \langle-\frac{d^2}{dx^2}\frac{u}{w}, v\rangle,$$

and so $\left[-\frac{1}{w(x)}\frac{d^2}{dx^2}\right]^* = \left[-\frac{d^2}{dx^2}\frac{1}{w(x)}\right]$. This is not the same unless $1/w$ commutes with the derivative, which only happens if w is a constant. More explicitly, let $\frac{1}{w} = c(x)$. Then

$$\hat{A}^*u = -\frac{d^2}{dx^2}(cu) = -c''u - 2c'u' - cu'',$$

which is $\neq -cu''$ for all $u(x)$ unless c' and c'' are zero, i.e. $c(x)$ is constant (and hence w is too).

(c) Suppose that V consists of functions with boundary conditions $u(0) = u(L) = 0$, and the inner product is $\langle u, v\rangle = \int_0^L w(x)u(x)v(x)dx$ for some function $w(x) > 0$.

(i) The first two follow by inspection: $\langle u, v\rangle = \int wuv = \int wvu = \langle v, u\rangle$, and $\langle \alpha u_1 + \beta u_2, v\rangle = \int w(\alpha u_1 + \beta u_2)v = \alpha \int wu_1v + \beta u_2v = \alpha\langle u_1, v\rangle + \beta\langle u_2, v\rangle$. Clearly, $\langle u, u\rangle = \int w|u|^2 \geq 0$, since $w > 0$ and $|u|^2 \geq 0$. Since the integrand is everywhere non-negative, the only way to have $\int w|u|^2 = 0$ is if $u(x) = 0$ except at isolated points covering zero area (technically, on a set of measure zero).

(ii) With this inner product, $\langle u, \hat{A}v\rangle = -\int u\frac{d^2}{dx^2}v$ (the w factors cancel), and then by integrating by parts twice as in class we obtain $-\int\left(\frac{d^2}{dx^2}u\right)v = \int w\left(-\frac{1}{w}\frac{d^2}{dx^2}u\right)v = \langle\hat{A}u, v\rangle$, hence $\hat{A}$ is symmetric. By the same argument, except $\langle u, \hat{A}u\rangle$ by parts only once, we obtain $\int |u'|^2 \geq 0$, which is $= 0$ only for $u = 0$ as in class, hence it is positive definite. Hence, as in class, the eigenvalues are real and positive, and the eigenfunctions are

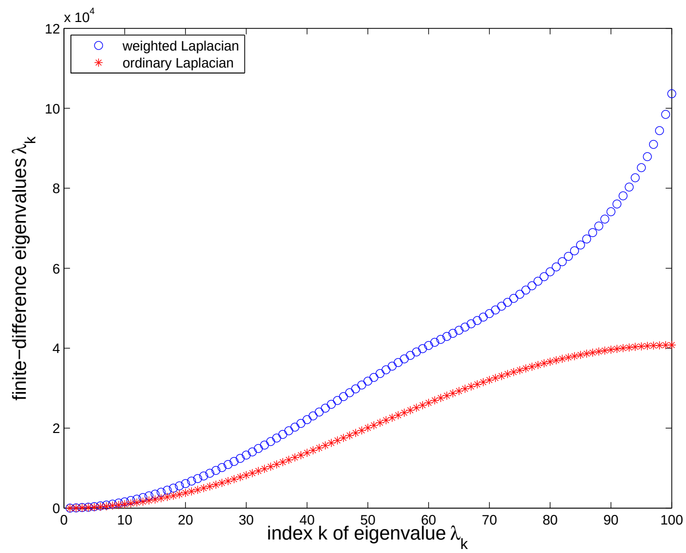

Figure 2: Plot of eigenvalues λ_k for the discretized $-e^x \frac{d^2}{dx^2}$ and $-\frac{d^2}{dx^2}$ operators, for problem 3(a).

orthogonal. And, as usual for real-symmetric operators, we expect that it is probably diagonalizable as long as $w(x)$ is not too crazy (diagonalizability only fails for weird non-physical operators with infinities somewhere, as mentioned in class, although defining this precisely takes a lot of functional analysis).

Problem 3: (5+5+5+10 points)

(a) There are 100 eigenvalues; rather than looking at the one by one, I'll check that they are real by asking Matlab for the maximum imaginary part: `max(abs(imag(diag(S))))`, which returns 0 as expected. The plot of λ is shown in Fig. 2. (If you had problems with plotting the two functions together, you could plot each one individually and use the `hold on` command in between to superimpose the plots.) Evidently, this $w(x)$ factor makes the eigenvalues bigger; this should not too surprising since we are multipling $\hat{A}$ by $e^x \geq 1$. For comparison, multiplying by a *constant* > 1 would just increase the eigenvalues by that factor, so it is not surprising that multiplying by a function ≥ 1 does something similar.

(b) These are plotted in Fig. 3. They are somewhat similar to $\sin(n\pi x/L)$ for $n = 1, 2, 3$, except that they oscillate faster towards the left-hand side (where w is larger).

(c) $u_1^T u_2 \approx 0.1762$, $u_1^T u_3 \approx -0.0276$, and $u_2^T u_3 \approx -0.1912$. These are far larger than mere roundoff errors, so the are definitely not orthogonal in the unweighted inner product.

(d) A discrete analogue of the weighted inner product from 2(c) is $\langle u, v\rangle = \sum_i w(x_i) u_i v_i$ (if we wanted to be picky we could multiply by Δx to make the sum approximate the integral, which in matlab is `u' * (v .* exp(-x))`. Using this inner product, I get $\langle u_1, u_2\rangle \approx -2.0610 \times 10^{-13}$, $\langle u_1, u_3\rangle \approx -2.4631 \times 10^{-14}$, and $\langle u_2, u_3\rangle \approx 1.2313 \times 10^{-13}$, which are $\neq 0$ but only because of roundoff errors; they are orthogonal to within the accuracy of the computer arithmetic.

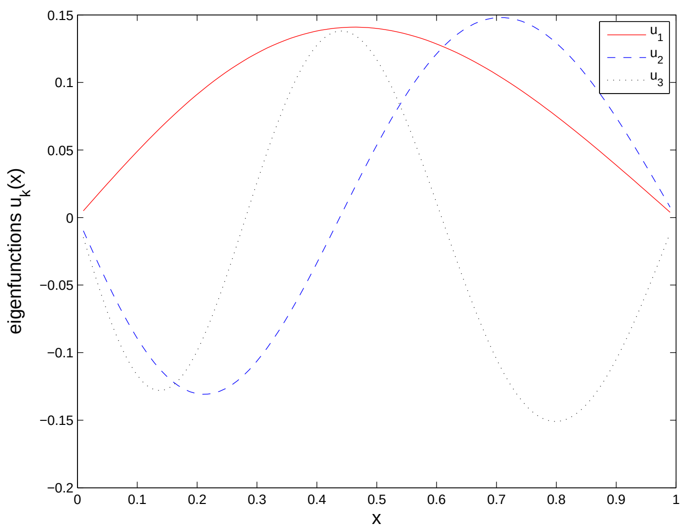

Figure 3: Plot of first three eigenfunctions of $e^x \frac{d^2}{dx^2}$ on $[0, 1]$ with zero boundary conditions.

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