18.303 Problem Set 1 Solutions

Problem 1: (10+10+10+5 points)

Here are a few questions that you should be able to answer based only on 18.06:

(a) For $A\mathbf{x} = \mathbf{b}$ to have a solution, $\mathbf{b}$ must be in the column space of A, which is orthogonal to the left nullspace, which is spanned by $\mathbf{m}$, so $\mathbf{b}^*\mathbf{m} = 0$ (or $\mathbf{b}^T\mathbf{m} = 0$ for real vectors) is true if and only if a solution exists. If a solution $\mathbf{x}$ exists, then all solutions are of the form $\mathbf{x} + \alpha\mathbf{n}$ for any scalar α, since $\mathbf{n}$ spans the nullspace.

(b) Solutions:

(i) A must be diagonalizable since it is $n \times n$ with n distinct eigenvalues. It need not be symmetric, however. (Real-symmetric implies real eigenvalues, but the converse is not true. For example, consider $A = V^{-1}\Sigma V$ where Σ is a diagonal matrix of the eigenvalues, and V is any real invertible matrix: A is similar to Σ and so has the same eigenvalues, but is not symmetric unless V is orthogonal.)

(ii) Since A is diagonalizable, its eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_{20}$ (corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{20} = -1, \ldots -20$) form a basis, and we can write $\mathbf{b} = \sum_i c_i \mathbf{v}_i$ for some coefficients c_i. Then $\mathbf{x}(t) = e^{At}\mathbf{b} = \sum_i e^{\lambda_i t} c_i \mathbf{v}_i$. Because all the λ_i are < 0, these terms are exponentially decaying and $\mathbf{x}(t) \to 0$ as $t \to \infty$. Moreover, for large finite $t \gg 1$, $\mathbf{x}(t)$ will be dominated by the slowest-decaying term (the largest λ_i with a $c_i \neq 0$). The largest eigenvalue is -1, and since $\mathbf{b}$ is random it is very unlikely that $c_1 \neq 0$, so we should have $\mathbf{x}(t) \approx e^{-t} c_1 \mathbf{v}_1$ for large t, i.e. $\mathbf{x}(t)$ will be approximately parallel to the eigenvector for $\lambda = -1$ and will be decaying approximately as e^{-t}.

(c) Let $\mathbf{x}^*$ denote the conjugate-transpose $\overline{\mathbf{x}^T}$. $A = A^T = \bar{A} = A^*$, and similarly for B. For an solution $A\mathbf{x} = \lambda B\mathbf{x}$, we therefore have

$$\begin{aligned} \mathbf{x}^* A\mathbf{x} &= \lambda(\mathbf{x}^* B\mathbf{x}) \\ &= (A\mathbf{x})^*\mathbf{x} = (\lambda B\mathbf{x})^*\mathbf{x} = \bar{\lambda}(\mathbf{x}^* B\mathbf{x}). \end{aligned}$$

Since B is positive-definite, this guarantees $\mathbf{x}^* B\mathbf{x} > 0$ for $\mathbf{x} \neq 0$ (zero is never counted as an eigevector),[1] so we can cancel it from both lines above, obtaining $\lambda = \bar{\lambda}$ and hence λ is real. If B was not given to be positive-definite, we could have $\mathbf{x}^* B\mathbf{x} = 0$, in which case it wouldn't cancel and λ might be complex.

(d) $\mathbf{z} = 2\mathbf{x} + 4\mathbf{y}$, since $A(2\mathbf{x} + 4\mathbf{y}) = 2A\mathbf{x} + 4A\mathbf{y} = 2\mathbf{b} + 4\mathbf{c}$. (This is sometimes called the "superposition" of two solutions, essentially another name for linearity.)

Problem 2: (10+10+5+10 points)

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

[1] Even if you only knew B was positive-definite for real $\mathbf{x}$, the complex positive-definiteness follows. Suppose $\mathbf{x}^T B\mathbf{x} > 0$ for any real $\mathbf{x} \neq 0$, with $B = B^T = \bar{B}$. Then, for any complex vector $\mathbf{z} = \mathbf{x} + i\mathbf{y} \neq 0$ for real $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{z}^* B\mathbf{z} = (\mathbf{x} - i\mathbf{y})^T B(\mathbf{x} + iy) = \mathbf{x}^T B\mathbf{x} + \mathbf{y}^T B\mathbf{y} - i\mathbf{y}^T B\mathbf{x} + i\mathbf{x}^T B\mathbf{y} > 0$, since the last two terms cancel ($\mathbf{y}^T B\mathbf{x} = \mathbf{x}^T B\mathbf{y}$ for $B = B^T$).

(a) All possible $f(x)$ do indeed form a vector space: this is just the column space of d^2/dx^2. More explicitly, recall that the column is a vector space because if $\frac{d^2}{dx^2}u_1(x) = f_1(x)$ and $\frac{d^2}{dx^2}u_2(x) = f_2(x)$ are possible right-hand sides, then clearly another possible right-hand side is $\alpha f_1 + \beta f_2 = \frac{d^2}{dx^2}(u_1 + u_2)$ for any scalars α and β . However, it is *not* the same as the vector space of the $u(x)$, because the boundary conditions are not the same. For example, consider $u(x) = x(L - x)$, which vanishes at the endpoints and hence is in the vector space. Then the corresponding function in the column space is $\frac{d^2}{dx^2}x(L - x) = -2$, which does not vanish at the endpoints.

(b) To have $\frac{d^2}{dx^2}u = \lambda u$, it still follows (as in class) that u is a sine, cosine, or exponential. Of these three possibilities, the only one with vanishing slope at $x = 0$ is cosine functions $\cos(kx)$ for some k. To have vanishing slope at $x = L$ as well, we must have L equal to an integer number of half-periods π/L, i.e. the eigenfunctions must be spanned by $\cos(n\pi x/L)$ for $n = 0, 1, 2, \ldots$. Note that $n = 0$ is now a valid eigenfunction, whereas before we excluded this (since sine of zero was zero).

(c) They do not form a vector space, since the space doesn't contain zero. Or, you could say that we cannot add, subtract, or multiply such functions by constants $\neq 1$ without the result violating the boundary condition $v(L) = 1$.

(d) Let $q(x)$ be any twice-differentiable function with $q(0) = 0$ and $q(L) = -1$, in which case $u(x) = v(x) + q(x)$ satisfies $u(0) = u(L) = 0$. For example, $q(x) = -x/L$ or $q(x) = -\sin(\pi x/2L)$ both work. Then substitute $v(x) = u(x) - q(x)$ into $\frac{d^2}{dx^2}v(x) = g(x)$ to obtain $\frac{d^2}{dx^2}u(x) = g(x) + \frac{d^2}{dx^2}q(x) = f(x)$. In the two examples mentioned previously, we obtain $f(x) = g(x)$ or $f(x) = g(x) + \left(\frac{\pi}{2L}\right)^2 \sin(x/2L)$, respectively. Thus, if we solve the $u(0) = u(L)$ Poisson equation with this new right-hand side $f(x)$, then we get back a solution $u(x) + q(x)$ to the $v(L) = 1$ equation!

Problem 3: (15+15 points)

(a) Now the null space consists of $u(x) = c$ for any constant c. This means that the Poisson solution, if it exists, is not unique: adding any constant yields another solution. Nor does the solution necessarily exist (even excluding crazy divergent right-hand-sides): from prolem 2(a), the eigenfunctions are now $\cos(n\pi x/L)$, so if we can express the right-hand-side as a cosine series $f(x) = \sum_n a_n \cos(n\pi x/L)$, a solution $u(x) = \sum_n \frac{a}{-(n\pi/L)^2} \cos(n\pi x/L)$ only exists of $a_0 = 0$, i.e. if $\int_0^L f(x)dx = 0$ (recalling the formula for the cosine series coefficients).

(b) Without any boundary conditions, null space is any cubic polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$. Imposing $u(0) = 0$ gives $a_0 = 0$. Imposing $u(L) = 0$ gives $a_1 = -a_2L + -a_3L^2$, so we only have two remaining degrees of freedom a_2 and a_3 (a two-dimensional null space). To make the null space $\{0\}$, it is sufficient to add the conditions $u'(0) = u'(L) = 0$. Setting $u'(0) = 0$ means that $a_1 = 0$ and hence $a_2 = -a_3L$. Setting $u'(L) = 0$ means that $2a_2 + 3a_3L = 0 = -2a_3L + 3a_3L = a_3L$, and hence $a_3 = 0$ and $a_2 = 0$. (It is still a vector space with these boundary conditions because they are zero: we can add, subtract, and multiply functions by constants while preserving the zero-slope property.)