

18.303 Problem Set 1

Due Wednesday, 15 September 2010.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

- Suppose A has a nullspace spanned by the vector \mathbf{n} , and a left nullspace spanned by the vector \mathbf{m} . In terms of \mathbf{m} , \mathbf{n} , \mathbf{b} , and/or \mathbf{x} only (*not* A), give an equation that will tell you whether $A\mathbf{x} = \mathbf{b}$ has a solution. Supposing it does have a solution \mathbf{x} , give an expression for *all* solutions.
- Suppose that A is a real 20×20 matrix with eigenvalues $-1, -2, \dots, -20$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{20}$, respectively.
 - Is A necessarily diagonalizable? Symmetric?
 - Suppose A is symmetric. If $\mathbf{x}(t)$ solves $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{b}$, for some random \mathbf{b} , what is likely to be true about the solution at large t ?
- In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A\mathbf{x} = \lambda\mathbf{x}$) must be real numbers if A is real-symmetric (if you forgot, look it up). Adapt the same proof to show that the solutions λ of $A\mathbf{x} = \lambda B\mathbf{x}$ must be real if A and B are real-symmetric and B is positive-definite.¹ Why does the proof fail if B is not positive-definite?
- Linearity means that if $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{c}$, then the solution \mathbf{z} of $A\mathbf{z} = 2\mathbf{b} + 4\mathbf{c}$ is $\mathbf{z} = ?$

Problem 2: Les Poissons, les Poissons, how I love les Poissons

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the “Dirichlet” boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

- Does the space of all possible right-hand sides $f(x)$ (for any u) form a vector space? Is it the same as the vector space of the $u(x)$ functions? Why or why not?
- How do the eigenfunctions of $\frac{d^2}{dx^2}$ change if we change the boundary conditions to $u'(0) = u'(L) = 0$?
- If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = 0, v(L) = 1$, do these functions form a vector space? Why or why not?
- Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L) = 0$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r . Suggest two different possible choices of q and r . (Transforming a new problem into an old, solved one is always a useful thing to do!)

¹The problem $A\mathbf{x} = \lambda B\mathbf{x}$ is known as a “generalized eigenproblem” and comes up a lot in numerical methods for PDE eigenproblems.

Problem 3: Nullicious

- (a) How does the null space of $\frac{d^2}{dx^2}$ differ if we consider the functions $u(x)$ on $x \in [0, L]$ with $u(0) = u(L) = 0$ (as in class) versus functions with different boundary conditions $u'(0) = u'(L) = 0$? What does this imply for the solutions of $\frac{d^2}{dx^2}u(x) = f(x)$, if any, in the latter case?
- (b) The linear operator $\frac{d^4}{dx^4}$ is important for something called the *beam equation* (for which Wikipedia has some pretty pictures). If we consider the vector space of functions $u(x)$ on $x \in [0, L]$ with the boundary conditions $u(0) = u(L) = 0$, then what is the null space of $\frac{d^4}{dx^4}$? Give an example of *additional* boundary conditions that you could add that shrink this null space to $\{0\}$ (but make sure it stays a vector space!).

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