Probability Topics: Independent & Mutually Exclusive Events*

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Abstract

This module explains the concept of independent events, where the probability of event A does not have any effect on the probability of event B, and mutually exclusive events, where events A and B cannot occur at the same time.

Independent and mutually exclusive do **not** mean the same thing.

1 Independent Events

Two events are independent if the following are true:

- P (A|B) = P (A)
- P (B|A) = P (B)
- P (A AND B) = P (A) · P (B)

If A and B are **independent**, then the chance of A occurring does not affect the chance of B occurring and vice versa. For example, two roles of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. To show two events are independent, you must show **only one** of the above conditions. If two events are NOT independent, then we say that they are **dependent**.

Sampling may be done **with replacement** or **without replacement**.

- **With replacement**: If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be independent, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement:**: When sampling is done without replacement, then each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be dependent or not independent.

If it is not known whether A and B are independent or dependent, **assume they are dependent until you can show otherwise**.

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2 Mutually Exclusive Events

A and B are **mutually exclusive** events if they cannot occur at the same time. This means that A and B do not share any outcomes and P(A AND B) = 0.

For example, suppose the sample space S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Let A = {1, 2, 3, 4, 5}, B = {4, 5, 6, 7, 8 and C = {7, 9}. A AND B = {4, 5}. P(A AND B) = $\frac{2}{10}$and is not equal to zero. Therefore, A and B are not mutually exclusive. A and C do not have any numbers in common so P(A AND C) = 0. Therefore, A and C are mutually exclusive.

If it is not known whether A and B are mutually exclusive, **assume they are not until you can show otherwise**.

The following examples illustrate these definitions and terms.

Example 1

Flip two fair coins. (This is an experiment.)

The sample space is {HH, HT, TH, TT} where T = tails and H = heads. The outcomes are HH, HT, TH, and TT. The outcomes HT and TH are different. The HT means that the first coin showed heads and the second coin showed tails. The TH means that the first coin showed tails and the second coin showed heads.

- Let A = the event of getting **at most one tail**. (At most one tail means 0 or 1 tail.) Then A can be written as {HH, HT, TH}. The outcome HH shows 0 tails. HT and TH each show 1 tail.
- Let B = the event of getting all tails. B can be written as {TT}. B is the **complement** of A. So, B = A'. Also, P (A) + P (B) = P (A) + P (A') = 1.
- The probabilities for A and for B are P (A) = $\frac{3}{4}$ and P (B) = $\frac{1}{4}$.
- Let C = the event of getting all heads. C = {HH}. Since B = {TT}, P (B AND C) = 0. B and C are mutually exclusive. (B and C have no members in common because you cannot have all tails and all heads at the same time.)
- Let D = event of getting **more than one** tail. D = {TT}. P (D) = $\frac{1}{4}$.
- Let E = event of getting a head on the first roll. (This implies you can get either a head or tail on the second roll.) $E = \{\text{HT}, \text{HH}\}$. P (E) = $\frac{2}{4}$.
- Find the probability of getting **at least one** (1 or 2) tail in two flips. Let F = event of getting at least one tail in two flips. $F = \{\text{HT}, \text{TH}, \text{TT}\}$. P(F) = $\frac{3}{4}$

Example 2

Roll one fair 6-sided die. The sample space is {1, 2, 3, 4, 5, 6}. Let event A = a face is odd. Then A = {1, 3, 5}. Let event B = a face is even. Then B = {2, 4, 6}.

- Find the complement of A, A'. The complement of A, A', is B because A and B together make up the sample space. P(A) + P(B) = P(A) + P(A') = 1. Also, P(A) = $\frac{3}{6}$ and P(B) = $\frac{3}{6}$
- Let event C = odd faces larger than 2. Then $C = \{3, 5\}$. Let event D = all even faces smaller than 5. Then $D = \{2, 4\}$. P(C and D) = 0 because you cannot have an odd and even face at the same time. Therefore, C and D are mutually exclusive events.
- Let event E = all faces less than 5. $E = \{1, 2, 3, 4\}$.

  Problem ***(Solution on p. 4.)***

  Are C and E mutually exclusive events? (Answer yes or no.) Why or why not?

- Find P(C|A). This is a conditional. Recall that the event C is {3, 5} and event A is {1, 3, 5}. To find P(C|A), find the probability of C using the sample space A. You have reduced the sample space from the original sample space {1, 2, 3, 4, 5, 6} to {1, 3, 5}. So, P(C|A) = $\frac{2}{3}$

Example 3

Let event G = taking a math class. Let event H = taking a science class. Then, G AND H = taking a math class and a science class. Suppose P(G) = 0.6, P(H) = 0.5, and P(G AND H) = 0.3. Are G and H independent?

If G and H are independent, then you must show **ONE** of the following:

- P(G|H) = P(G)
- P(H|G) = P(H)
- P(G AND H) = P(G) · P(H)

NOTE: **The choice you make depends on the information you have.** You could choose any of the methods here because you have the necessary information.

Problem 1

Show that P(G|H) = P(G).

Solution

$\mathrm{P(G|H)} = \frac{\mathrm{P(G\ AND\ H)}}{\mathrm{P(H)}} = \frac{0.3}{0.5} = 0.6 = \mathrm{P(G)}$

Problem 2

Show P(G AND H) =P(G) · P(H).

Solution

$\mathrm{P(G)} \cdot \mathrm{P(H)} = 0.6 \cdot 0.5 = 0.3 = \mathrm{P(G\ AND\ H)}$

Since G and H are independent, then, knowing that a person is taking a science class does not change the chance that he/she is taking math. If the two events had not been independent (that is, they are dependent) then knowing that a person is taking a science class would change the chance he/she is taking math. For practice, show that P(H|G) = P(H) to show that G and H are independent events.

Example 4

In a box there are 3 red cards and 5 blue cards. The red cards are marked with the numbers 1, 2, and 3, and the blue cards are marked with the numbers 1, 2, 3, 4, and 5. The cards are well-shuffled. You reach into the box (you cannot see into it) and draw one card.

Let R = red card is drawn, B = blue card is drawn, E = even-numbered card is drawn.

The sample space S = R1, R2, R3, B1, B2, B3, B4, B5. S has 8 outcomes.

- P(R) = $\frac{3}{8}$. P(B) = $\frac{5}{8}$. P(R AND B) = 0. (You cannot draw one card that is both red and blue.)
- P(E) = $\frac{3}{8}$. (There are 3 even-numbered cards, R2, B2, and B4.)
- P(E|B) = $\frac{2}{5}$. (There are 5 blue cards: B1, B2, B3, B4, and B5. Out of the blue cards, there are 2 even cards: B2 and B4.)
- P(B|E) = $\frac{2}{3}$. (There are 3 even-numbered cards: R2, B2, and B4. Out of the even-numbered cards, 2 are blue: B2 and B4.)
- The events R and B are mutually exclusive because P(R AND B) = 0.
- Let G = card with a number greater than 3. $G = \{\mathrm{B4, B5}\}$. P(G) = $\frac{2}{8}$. Let H = blue card numbered between 1 and 4, inclusive. $H = \{\mathrm{B1, B2, B3, B4}\}$. P(G|H) = $\frac{1}{4}$. (The only card in H that has a number greater than 3 is B4.) Since $\frac{2}{8} = \frac{1}{4}$, P(G) = P(G|H) which means that G and H are independent.

Solutions to Exercises in this Module

Solution to Example 2, Problem (p. 2)
No. $C = \{3,\ 5\}$ and $E = \{1,\ 2,\ 3,\ 4\}$. $\mathrm{P}\left(\mathrm{C\ AND\ E}\right) = \frac{1}{6}$. To be mutually exclusive, $\mathrm{P}\left(\mathrm{C\ AND\ E}\right)$ must be 0.

Glossary

Definition 1: Independent Events
The occurrence of one event has no effect on the probability of the occurrence of any other event. Events A and B are independent if one of the following is true: (1). $P\left(A|B\right) = P\left(A\right)$; (2) $P\left(B|A\right) = P\left(B\right)$; (3) $P\left(A\text{and}B\right) = P\left(A\right)P\left(B\right)$.

Definition 2: Mutually Exclusive
An observation cannot fall into more than one class (category). Being in more than one category prevents being in a mutually exclusive category.