GROWTH AND DECAY

In this set of supplemental notes, I will provide more worked examples of a type of differential equations that their solutions are exponential functions. These kinds of problems can represent the exponential growth or decay of a substance. I will first state the law of exponential change.

Law of Exponential Change

\[ y = y_0 e^{kt} \]

\(y_0\) is the initial amount of the substance present at \(t = 0\). \(k\) is the rate constant. If \(k > 0\), then it is a growth constant. If \(k < 0\), then it is a decay constant.

**EXAMPLE 1:**

The earth's atmospheric pressure \(p\) is often modeled by assuming that the rate \(dp/dh\) at which \(p\) changes with the altitude \(h\) above sea level is proportional to \(p\). Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

a. Solve the initial value problem.

\[ \text{DE: } \frac{dp}{dh} = kp \quad \text{IC: } p = p_0 \text{ when } h = 0 \]

\(k\) is a constant

**SOLUTION:**

\[ \frac{dp}{dh} = kp \quad \rightarrow \quad \frac{dp}{p} = k\, dh \]

\[ \int \frac{dp}{p} = \int k\, dh \]

\[ \ln p = kh + C \]
\[ p = e^{kh + C} \]
\[ p = e^{kh} e^C \]
\[ p = Ae^{kh} \]

At sea level (\( h = 0 \)), \( p_0 = 1013 \) millibars, and when \( h = 20 \) km, \( p = 90 \) millibars. Using this data, let us first solve for \( A \).

\[ 1013 = Ae^{0} \text{ or } A = 1013 \]

Now \( p = 1013e^{kh} \).

To determine the value for \( k \) we will use the second set of data.

\[ 90 = 1013e^{20k} \implies 0.088845015 = e^{20k} \implies \ln 0.088845015 = 20k \implies k = -0.121043092 \]

b. What is the atmospheric pressure at \( h = 50 \) km?

**SOLUTION:**

We will solve for \( p \) when \( h = 50 \).

\[ p = 1013e^{-0.121043092(50)} \Rightarrow p = 2.383373495 \text{ millibars} \]

c. At what altitude does the pressure equal 900 millibars?

**SOLUTION:**

We will solve for \( h \) when \( p = 900 \) millibars.

\[ 900 = 1013e^{-0.121043092h} \implies 0.888450148 = e^{-0.121043092h} \implies \ln 0.888450148 = -0.121043092h \implies h = 0.977145733 \text{ km} \]

**EXAMPLE 2:**

Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage \( V \) across its terminals and that, if \( t \) is measured in seconds,
\[ \frac{dV}{dt} = -\frac{1}{40} V. \]

Solve this equation for \( V \), using \( V_0 \) to denote the value of \( V \) when \( t = 0 \). How long will it take the voltage to drop to 10% of its original value?

**SOLUTION:**

We will first solve the above initial value problem.

\[
\frac{dV}{dt} = -\frac{1}{40} V \quad \Rightarrow \quad \frac{dV}{V} = -\frac{1}{40} \, dt
\]

\[
\int \frac{dV}{V} = -\frac{1}{40} \int dt \quad \Rightarrow \quad \ln V = -\frac{1}{40} t + C
\]

\[
V = e^{-\frac{1}{40} t + C}
\]

\[
= e^{-\frac{1}{40} t} \cdot e^C
\]

\[
V = Ae^{-\frac{1}{40} t}
\]

When \( t = 0, \) \( V = V_0 \), so let us determine the value for \( A \).

\[
V_0 = Ae^{-\frac{1}{40} (0)} \quad \Rightarrow \quad A = V_0
\]

Therefore, the equation is the following.

\[
V = V_0 e^{-\frac{1}{40} t}
\]

Now, we want to determine how long it will take the voltage to drop to 10% of its original value. To do this, we will solve the following equation for \( t \).
\[0.1V_0 = V_0 e^{-\frac{1}{40} t} \quad \Rightarrow \quad 0.1 = e^{-\frac{1}{40} t}\]

\[\ln 0.1 = -\frac{1}{40} t \quad \Rightarrow \quad -2.302585093 = -\frac{1}{40} t\]

\[t = 92.10340372 \text{ seconds}\]

**EXAMPLE 3:**

Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decrease at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

**SOLUTION:**

Using the law of exponential change, \(y = y_0 e^{kt}\), and the fact that when \(t = 0\), \(A = A_0\), the equation becomes \(A = A_0 e^{kt}\). We are given that the amount decreases at a continuous rate of 10% per year. So when \(t = 1\) year, \(A = .9A_0\). We will use these two values to find \(k\).

\[0.9A_0 = A_0 e^k \rightarrow 0.9 = e^k \rightarrow \ln 0.9 = k \text{ or } k = -0.105360516\]

Now to determine when there will be one-fifth of the present amount will be left.

\[(1/5)A_0 = A_0 e^{-0.105360516 t} \rightarrow 0.2 = e^{-0.105360516 t}\]

\[\rightarrow \ln 0.2 = -0.105360516 t \rightarrow t = 15.27553179 \text{ years}\]

**EXAMPLE 4:**

The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

**SOLUTION:**

The half-life of carbon-14 is 5700 years. The decay constant, \(k\), is defined to be
Let $A_0$ be the amount of carbon-14 that is found in living matter ($t = 0$ years). We want to determine when there is $44.5\%$ of $A_0$ is left.

$$0.445A_0 = A_0 e^{-\frac{\ln 2}{5700}t}$$

$$0.445 = e^{-\frac{\ln 2}{5700}t}$$

$$\ln 0.445 = -\frac{\ln 2}{5700}t$$

$$t = 6658.299725 \text{ years}$$

**EXAMPLE 5:**

A painting attributed to Vermeer (1632 - 1675), which should contain no more than $96.2\%$ of its original carbon-14, contains $99.5\%$ instead. About how old is the forgery?

**SOLUTION:**

Using the value for $k$ from example 4, and $A = A_0$ when $t = 0$, we will solve the following equation for $t$.

$$0.995A_0 = A_0 e^{-\frac{\ln 2}{5700}t}$$

$$0.995 = e^{-\frac{\ln 2}{5700}t}$$

$$\ln 0.995 = -\frac{\ln 2}{5700}t$$

$$t = 41.22 \text{ years}$$
The forgery is only 41.22 years old.

Work through these examples. If you have any questions or problems with any of these examples, feel free to contact me.