Nilpotent group

In mathematics, more specifically in the field of group theory, a **nilpotent group** is a group that is "almost abelian". This idea is motivated by the fact that nilpotent groups are solvable, and for finite nilpotent groups, two elements having relatively prime orders must commute. It is also true that finite nilpotent groups are supersolvable.

Nilpotent groups arise in Galois theory, as well as in the classification of groups. They also appear prominently in the classification of Lie groups.

Analogous terms are used for Lie algebras (using the Lie bracket) including **nilpotent**, **lower central series**, and **upper central series**.

Definition

The definition uses the idea, explained on its own page, of a central series for a group. The following are equivalent formulations:

- A nilpotent group is one that has a central series of finite length.
- A nilpotent group is one whose lower central series terminates in the trivial subgroup after finitely many steps.
- A nilpotent group is one whose upper central series terminates in the whole group after finitely many steps.

For a nilpotent group, the smallest n such that G has a central series of length n is called the **nilpotency class** of G ; and G is said to be **nilpotent of class *n***. (By definition, the length is n if there are $n + 1$ different subgroups in the series, including the trivial subgroup and the whole group.)

Equivalently, the nilpotency class of G equals the length of the lower central series or upper central series. If a group has nilpotency class at most m, then it is sometimes called a **nil-*m* group**.

It follows immediately from any of the above forms of the definition of nilpotency, that the trivial group is the unique group of nilpotency class 0, and groups of nilpotency class 1 are exactly the non-trivial abelian groups.[1] [2]

Examples

- As noted above, every abelian group is nilpotent.[1] [3]
- For a small non-abelian example, consider the quaternion group Q_8, which is a smallest non-abelian p-group. It has center $\{1, -1\}$ of order 2, and its upper central series is $\{1\}$, $\{1, -1\}$, Q_8; so it is nilpotent of class 2.
- All finite p-groups are in fact nilpotent (proof). The maximal class of a group of order p^n is n - 1. The 2-groups of maximal class are the generalised quaternion groups, the dihedral groups, and the semidihedral groups.
- The direct product of two nilpotent groups is nilpotent.[4]
- Conversely, every finite nilpotent group is the direct product of p-groups.[5]
- The Heisenberg group is an example of non-abelian,[6] infinite nilpotent group.[7]
- The multiplicative group of upper unitriangular n x n matrices over any field F is a nilpotent group of nilpotent length n - 1 .
- The multiplicative group of invertible upper triangular n x n matrices over a field F is not in general nilpotent, but is solvable.

Explanation of term

Nilpotent groups are so called because the "adjoint action" of any element is nilpotent, meaning that for a nilpotent group G of nilpotence degree n and an element g, the function $\operatorname{ad}_g\colon G \to G$ defined by $\operatorname{ad}_g(x) := [g,x]$ (where $[g,x] = g^{-1}x^{-1}gx$ is the commutator of g and x) is nilpotent in the sense that the nth iteration of the function is trivial: $\left(\operatorname{ad}_g\right)^n(x) = e$ for all x in G .

This is not a defining characteristic of nilpotent groups: groups for which ad_g is nilpotent of degree n (in the sense above) are called n-Engel groups,[8] and need not be nilpotent in general. They are proven to be nilpotent if they have finite order, and are conjectured to be nilpotent as long as they are finitely generated.

An abelian group is precisely one for which the adjoint action is not just nilpotent but trivial (a 1-Engel group).

Properties

Since each successive factor group Z_{i+1}/Z_i in the upper central series is abelian, and the series is finite, every nilpotent group is a solvable group with a relatively simple structure.

Every subgroup of a nilpotent group of class n is nilpotent of class at most n;[9] in addition, if f is a homomorphism of a nilpotent group of class n, then the image of f is nilpotent[9] of class at most n.

The following statements are equivalent for finite groups,[10] revealing some useful properties of nilpotency:

- G is a nilpotent group.
- If H is a proper subgroup of G, then H is a proper normal subgroup of $N_G(H)$ (the normalizer of H in G). This is called the **normalizer property** and can be phrased simply as "normalizers grow".
- Every maximal proper subgroup of G is normal.
- G is the direct product of its Sylow subgroups.

The last statement can be extended to infinite groups: if G is a nilpotent group, then every Sylow subgroup G_p of G is normal, and the direct product of these Sylow subgroups is the subgroup of all elements of finite order in G (see torsion subgroup).

Many properties of nilpotent groups are shared by hypercentral groups.

References

[1] Suprunenko (1976), p. 205 (http://books.google.com/books?id=cTtuPOj5h10C&pg=PA205&dq="abelian+group+is+nilpotent")

[2] Tabachnikova & Smith (2000), p. 169 (http://books.google.com/books?id=DD0TW28WjfQC&pg=PA169&dq="The+trivial+group+has+nilpotency+class+0")

[3] Hungerford (1974), p. 100 (http://books.google.com/books?id=t6N_tOQhafoC&pg=PA100&dq="every+abelian+group+G+is+nilpotent")

[4] Zassenhaus (1999), p. 143 (http://books.google.com/books?id=eCBK6tj7_vAC&pg=PA143&dq="The+direct+product+of+a+finite+number+of+nilpotent+groups+is+nilpotent")

[5] Zassenhaus (1999), p. 143, Theorem 11 (http://books.google.com/books?id=eCBK6tj7_vAC&pg=PA143&dq="Every+finite+nilpotent+group+is+the+direct+product+of+its+Sylow+groups")

[6] Haeseler (2002), p. 15 (http://books.google.com/books?id=wmh7tc6uGosC&pg=PA15&dq="The+Heisenberg+group+is+a+non-abelian")

[7] Palmer (2001), p. 1283 (http://books.google.com/books?id=zn-iZNNTb-AC&pg=PA1283&dq="Heisenberg+group+this+group+has+nilpotent+length+2+but+is+not+abelian")

[8] For the term, compare Engel's theorem, also on nilpotency.

[9] Bechtell (1971), p. 51, Theorem 5.1.3

[10] Isaacs (2008), Thm. 1.26

- *Homology in group theory*, by Urs Stammbach, Lecture Notes in Mathematics, Volume 359, Springer-Verlag, New York, 1973, vii+183 pp. review (http://projecteuclid.org/euclid.bams/1183537230)
- Suprunenko, D. A. (1976). *Matrix Groups*. Providence, Rhode Island: American Mathematical Society. ISBN 0-8218-1341-2.

- Hungerford, Thomas Gordon (1974). *Algebra*. Berlin: Springer-Verlag. ISBN 0-387-90518-9.
- Palmer, Theodore W. (1994). *Banach algebras and the general theory of *-algebras*. Cambridge, UK: Cambridge University Press. ISBN 0-521-36638-0.
- Friedrich Von Haeseler (2002). *Automatic Sequences (De Gruyter Expositions in Mathematics, 36)*. Berlin: Walter de Gruyter. ISBN 3-11-015629-6.
- Isaacs, I. Martin (2008). *Finite group theory*. American Mathematical Society. ISBN 0-8218-4344-3.
- Zassenhaus, Hans (1999). *The theory of groups*. New York: Dover Publications. ISBN 0-486-40922-8.
- Bechtell, Homer (1971). *The theory of groups*. Addison-Wesley.
- Tabachnikova, Olga; Smith, Geoff (2000). *Topics in Group Theory (Springer Undergraduate Mathematics Series)*. Berlin: Springer. ISBN 1-85233-235-2.

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