PARAMETERIZATIONS OF PLANE CURVES

Suppose we want to plot the path of a particle moving in a plane. This path looks like a curve, but we cannot plot it like we would plot any other type of curve in the Cartesian plane. The reason for this is the fact that we cannot express \( y \) directly in terms of \( x \) or \( x \) in terms of \( y \). To get around this problem, we can describe the path of the particle with a pair of equations, \( x = f(t) \) and \( y = g(t) \).

**FACT:** If \( x \) and \( y \) are given as continuous functions \( x = f(t) \), \( y = g(t) \) over an interval of \( t \)-values, then the set of points \( (x, y) = (f(t), g(t)) \) defined by these equations is a curve in the coordinate plane. The equations are parametric equations for the curve. The variable \( t \) is a parameter for the curve and its domain \( I \) is the parameter interval \( I \). If \( I \) is a closed interval, \( a \leq t \leq b \), the point \( (f(a), g(a)) \) is the initial point of the curve, and \( (f(b), g(b)) \) is the terminal point of the curve.

Here are some parameterizations that are used frequently.

**CIRCLE:** \( x^2 + y^2 = a^2 \)  
\[ x = a \cos t \quad 0 \leq t \leq 2\pi \]  
\[ y = a \sin t \]

**ELLIPSE:**  
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]  
\[ x = a \cos t \quad 0 \leq t \leq 2\pi \]  
\[ y = b \sin t \]

**FINDING THE PARAMETRIC EQUATIONS FROM THE CARTESIAN EQUATIONS**

**EXAMPLE 1:** Parameterize \( y = x^3 - 4x^2 + 6 \).

**SOLUTION:** Since the given equation has \( y \) in terms of \( x \), let \( x = t \).

Now substitute \( t \) in for \( x \) into the original equation.

\( y = t^3 - 4t^2 + 6 \)

Therefore, the parametric equations for the given equation are the following.

\[ x = t \quad y = t^3 - 4t^2 + 6 \quad -\infty < t < \infty \]

Here is the graph using these parametric equations.
EXAMPLE 2: Fin the parametric equations for $x^2 - y^2 = 1$.

SOLUTION: What trig function do we know that has the property $x^2 - y^2 = 1$? Well, let us see.

$\cos^2 t + \sin^2 t = 1$ Nope!

$\tan^2 t + 1 = \sec^2 t \rightarrow \sec^2 t - \tan^2 t = 1$

This is the one we want to use. So let $x = \sec t$ and $y = \tan t$, and let the parameterization interval be the interval $-\infty < t < \infty$. Here is the graph of this set of parametric equations.
EXAMPLE 3: Find the parametric equations for \( x = y^2 - 4y + 3 \).

SOLUTION: Since \( x \) is in terms of \( y \), let \( y = t \). Then \( x = t^2 - 4t + 3 \).

Therefore the parametric equations are \( x = t^2 - 4t + 3 \) and \( y = t \), and the parametric interval is \(-\infty < t < \infty \). Here is the graph of this set of parametric equations.

EXAMPLE 4: Find the parametric equations for the following equation.

\[
\frac{x^2}{16} + \frac{y^2}{9} = 1
\]

SOLUTION: This is an equation of an ellipse, so it is one of the common parameterizations. Let \( a = 4 \) and \( b = 3 \), so the parameterization of this ellipse is \( x = 4 \cos \, t \) and \( y = 3 \sin \, t \). The parametric interval for this parameterization is \( 0 \leq t \leq 2\pi \). Here is the graph of this parameterization.
FINDING THE CARTESIAN EQUATIONS FROM PARAMETRIC EQUATIONS

**EXAMPLE 5:** Find the Cartesian equation for \( x = \cos 4t, \ y = \sin 4t, \ 0 \leq t \leq \pi / 2. \)

**SOLUTION:** Anytime the parameterization involves trigonometric functions, use one of the trig identities to put it back into Cartesian form.

We know that \( \sin^2 t + \cos^2 t = 1, \) or in our case, \( \sin^2 4t + \cos^2 4t = 1. \)

This implies that \( y^2 + x^2 = 1, \) and the equation is a circle of radius 1.

Here is the graph of this set of parametric equations.
EXAMPLE 6: Find the Cartesian equation for \( x = 3t, \ y = 9t^2, \ -\infty < t < \infty \).

SOLUTION: First of all, solve for \( t \) in terms of \( x \).

\[
x = 3t \quad \rightarrow \quad t = \frac{x}{3}
\]

Now sub that expression in the \( y \) equation for \( t \).

\[
y = 9t^2 \quad \rightarrow \quad y = 9\left(\frac{x}{3}\right)^2 \quad \rightarrow \quad y = x^2
\]

Here is the graph of this set of parametric equations.
EXAMPLE 7: Find the Cartesian equation for $x = \sec^2 t - 1$, $y = \tan t$, $-\pi / 2 < t < \pi/2$.

SOLUTION: We know that $\tan^2 t + 1 = \sec^2 t$, therefore $\tan^2 t = \sec^2 t - 1$.

This implies that $y^2 = x$.

Here is the graph of this set of parametric equations.

EXAMPLE 8: Find the Cartesian equation for $x = 3 - 3t$, $y = 2t$, $0 \leq t \leq 1$.

SOLUTION: First of all, let us solve $t$ in terms of $y$.

$$y = 2t \rightarrow \frac{y}{2} = t$$
Now sub this expression into $t$ in the $x$ equation.

$$x = 3 - 3t \quad \rightarrow \quad x = 3 - 3\left(\frac{y}{2}\right) \quad \rightarrow \quad 2x = 6 - 3y \quad \text{or} \quad 2x + 3y = 6$$

Here is the graph of this set of parametric equations.

EXAMPLE 9: Find the Cartesian equation for the following set of parametric equations.

$$x = t \quad y = \sqrt{4 - t^2} \quad 0 \leq t \leq 2$$

SOLUTION: Substitute $x$ in for $t$ in the $y$ equation.

$$y = \sqrt{4 - t^2} \quad \rightarrow \quad y = \sqrt{4 - x^2}$$

Here is the graph of this set of parametric equations.
As you can see, finding parameterization for Cartesian equations is a straightforward process. Also going from a parameterization to a Cartesian equation is not that bad either. If you want to graph these equations on your calculator, all you have to do is to make sure that your calculator mode is set to parametric instead of function.