General linear group

In mathematics, the **general linear group** of degree n is the set of $n{\times}n$ invertible matrices, together with the operation of ordinary matrix multiplication. This forms a group, because the product of two invertible matrices is again invertible, and the inverse of an invertible matrix is invertible. The name is because the columns of an invertible matrix are linearly independent, hence the vectors/points they define are in general linear position, and matrices in the general linear group take points in general linear position to points in general linear position.

To be more precise, it is necessary to specify what kind of objects may appear in the entries of the matrix. For example, the general linear group over **R** (the set of real numbers) is the group of $n{\times}n$ invertible matrices of real numbers, and is denoted by $GL_n(\mathbf{R})$ or $GL(n, \mathbf{R})$.

More generally, the general linear group of degree n over any field F (such as the complex numbers), or a ring R (such as the ring of integers), is the set of $n{\times}n$ invertible matrices with entries from F (or R), again with matrix multiplication as the group operation.[1] Typical notation is $GL_n(F)$ or $GL(n, F)$, or simply $GL(n)$ if the field is understood.

More generally still, the general linear group of a vector space $GL(V)$ is the abstract automorphism group, not necessarily written as matrices.

The **special linear group**, written $SL(n, F)$ or $SL_n(F)$, is the subgroup of $GL(n, F)$ consisting of matrices with a determinant of 1.

The group $GL(n, F)$ and its subgroups are often called **linear groups** or **matrix groups** (the abstract group $GL(V)$ is a linear group but not a matrix group). These groups are important in the theory of group representations, and also arise in the study of spatial symmetries and symmetries of vector spaces in general, as well as the study of polynomials. The modular group may be realised as a quotient of the special linear group SL(2, **Z**).

If $n \geq 2$, then the group $GL(n, F)$ is not abelian.

General linear group of a vector space

If V is a vector space over the field F, the general linear group of V, written GL(V) or Aut(V), is the group of all automorphisms of V, i.e. the set of all bijective linear transformations $V \to V$, together with functional composition as group operation. If V has finite dimension n, then GL(V) and GL(n, F) are isomorphic. The isomorphism is not canonical; it depends on a choice of basis in V. Given a basis $(e_1, ..., e_n)$ of V and an automorphism T in GL(V), we have

$$Te_k = \sum_{j=1}^{n} a_{jk} e_j$$

for some constants a_{jk} in F; the matrix corresponding to T is then just the matrix with entries given by the a_{jk}.

In a similar way, for a commutative ring R the group GL(n, R) may be interpreted as the group of automorphisms of a *free* R-module M of rank n. One can also define GL(M) for any R-module, but in general this is not isomorphic to GL(n, R) (for any n).

In terms of determinants

Over a field *F*, a matrix is invertible if and only if its determinant is nonzero. Therefore an alternative definition of GL(*n*, *F*) is as the group of matrices with nonzero determinant.

Over a commutative ring *R*, one must be slightly more careful: a matrix over *R* is invertible if and only if its determinant is a unit in *R*, that is, if its determinant is invertible in *R*. Therefore GL(*n*, *R*) may be defined as the group of matrices whose determinants are units.

Over a non-commutative ring *R*, determinants are not at all well behaved. In this case, GL(*n*, *R*) may be defined as the unit group of the matrix ring M(*n*, *R*).

As a Lie group

Real case

The general linear group GL(*n*,**R**) over the field of real numbers is a real Lie group of dimension n^2. To see this, note that the set of all *n*×*n* real matrices, $M_n(\mathbf{R})$, forms a real vector space of dimension n^2. The subset GL(*n*,**R**) consists of those matrices whose determinant is non-zero. The determinant is a polynomial map, and hence GL(*n*,**R**) is a open affine subvariety of $M_n(\mathbf{R})$ (a non-empty open subset of $M_n(\mathbf{R})$ in the Zariski topology), and therefore[2] a smooth manifold of the same dimension.

The Lie algebra of GL(*n*,**R**), denoted $\mathfrak{gl}_n$, consists of all *n*×*n* real matrices with the commutator serving as the Lie bracket.

As a manifold, GL(*n*,**R**) is not connected but rather has two connected components: the matrices with positive determinant and the ones with negative determinant. The identity component, denoted by $GL^+(n, \mathbf{R})$, consists of the real *n*×*n* matrices with positive determinant. This is also a Lie group of dimension n^2; it has the same Lie algebra as GL(*n*,**R**).

The group GL(*n*,**R**) is also noncompact. "The"[3] maximal compact subgroup of GL(*n*, **R**) is the orthogonal group O(*n*), while "the" maximal compact subgroup of $GL^+(n, \mathbf{R})$ is the special orthogonal group SO(*n*). As for SO(*n*), the group $GL^+(n, \mathbf{R})$ is not simply connected (except when *n*=1), but rather has a fundamental group isomorphic to **Z** for *n*=2 or $\mathbf{Z}_2$ for *n*>2.

Complex case

The general linear GL(*n*,**C**) over the field of complex numbers is a *complex* Lie group of complex dimension n^2. As a real Lie group it has dimension $2n^2$. The set of all real matrices forms a real Lie subgroup. These correspond to the inclusions

  GL(*n*,**R**) < GL(*n*,**C**) < GL(*2n*,**R**),

which have real dimensions n^2, $2n^2$, and $4n^2 = (2n)^2$. Complex *n*-dimensional matrices can be characterized as real 2*n*-dimensional matrices that preserve a linear complex structure − concretely, that commute with a matrix *J* such that $J^2 = -I$, where *J* corresponds to multiplying by the imaginary unit *i*.

The Lie algebra corresponding to GL(*n*,**C**) consists of all *n*×*n* complex matrices with the commutator serving as the Lie bracket.

Unlike the real case, GL(*n*,**C**) is connected. This follows, in part, since the multiplicative group of complex numbers **C*** is connected. The group manifold GL(*n*,**C**) is not compact; rather its maximal compact subgroup is the unitary group U(*n*). As for U(*n*), the group manifold GL(*n*,**C**) is not simply connected but has a fundamental group isomorphic to **Z**.

Over finite fields

If F is a finite field with q elements, then we sometimes write GL(n, q) instead of GL(n, F). When p is prime, GL(n, p) is the outer automorphism group of the group $Z_p^{\ n}$, and also the automorphism group, because $Z_p^{\ n}$ is Abelian, so the inner automorphism group is trivial.

The order of GL(n, q) is:

$$(q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{n-1})$$

This can be shown by counting the possible columns of the matrix: the first column can be anything but the zero vector; the second column can be anything but the multiples of the first column; and in general, the kth column can be any vector not in the linear span of the first $k - 1$ columns. In q-analog notation, this is $[n]_q!(q-1)^n$.

Cayley table of GL(2,2), which is isomorphic to S_3.

For example, GL(3, 2) has order $(8 - 1)(8 - 2)(8 - 4) = 168$. It is the automorphism group of the Fano plane and of the group $Z_2^{\ 3}$, and is also known as PSL(2,7).

More generally, one can count points of Grassmannian over F: in other words the number of subspaces of a given dimension k. This requires only finding the order of the stabilizer subgroup of one such subspace (described on that page in block matrix form), and dividing into the formula just given, by the orbit-stabilizer theorem.

These formulas are connected to the Schubert decomposition of the Grassmannian, and are q-analogs of the Betti numbers of complex Grassmannians. This was one of the clues leading to the Weil conjectures.

Note that in the limit as $q \to 1,$ the order of GL(n, q) goes to $n!$ which is the order of the symmetric group − in the philosophy of the field with one element, one thus interprets the symmetric group as the general linear group over the field with one element: $S_n \cong GL(n, 1)$.

History

The general linear group over a prime field, GL(v,p), was constructed and its order computed by Évariste Galois in 1832, in his last letter (to Chevalier) and second (of three) attached manuscripts, which he used in the context of studying the Galois group of the general equation of order p^v.[4]

Special linear group

The special linear group, SL(n, F), is the group of all matrices with determinant 1. They are special in that they lie on a subvariety − they satisfy a polynomial equation (as the determinant is a polynomial in the entries). Matrices of this type form a group as the determinant of the product of two matrices is the product of the determinants of each matrix. SL(n, F) is a normal subgroup of GL(n, F).

If we write $F^\times$ for the multiplicative group of F (excluding 0), then the determinant is a group homomorphism

$$\det: GL(n, F) \to F^\times.$$

The kernel of the map is just the special linear group. By the first isomorphism theorem we see that GL(n,F)/SL(n,F) is isomorphic to $F^\times$. In fact, GL(n, F) can be written as a semidirect product of SL(n, F) by $F^\times$:

$$GL(n, F) = SL(n, F) \rtimes F^\times$$

When F is **R** or **C**, SL(n) is a Lie subgroup of GL(n) of dimension $n^2 - 1$. The Lie algebra of SL(n) consists of all $n \times n$ matrices over F with vanishing trace. The Lie bracket is given by the commutator.

The special linear group SL(n, **R**) can be characterized as the group of *volume and orientation preserving* linear transformations of $\mathbf{R}^n$.
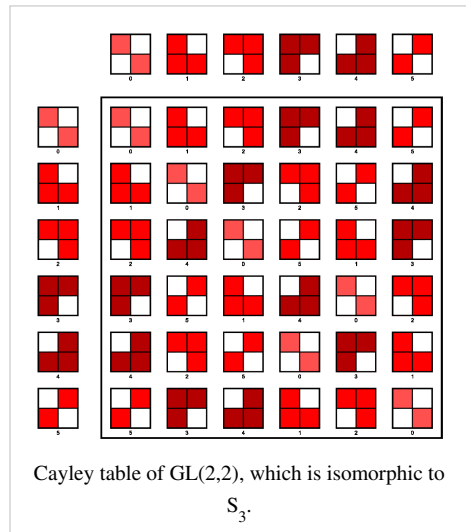
The group SL(n, **C**) is simply connected while SL(n, **R**) is not. SL(n, **R**) has the same fundamental group as GL^{+}(n, **R**), that is, **Z** for n=2 and **Z**$_2$ for n>2.

Other subgroups

Diagonal subgroups

The set of all invertible diagonal matrices forms a subgroup of GL(n, F) isomorphic to $(F^{\times})^n$. In fields like **R** and **C**, these correspond to rescaling the space; the so called dilations and contractions.

A **scalar matrix** is a diagonal matrix which is a constant times the identity matrix. The set of all nonzero scalar matrices forms a subgroup of GL(n, F) isomorphic to $F^{\times}$. This group is the center of GL(n, F). In particular, it is a normal, abelian subgroup.

The center of SL(n, F) is simply the set of all scalar matrices with unit determinant, and is isomorphic to the group of nth roots of unity in the field F.

Classical groups

The so-called *classical groups* are subgroups of GL(V) which preserve some sort of bilinear form on a vector space V. These include the

- **orthogonal group**, O(V), which preserves a non-degenerate quadratic form on V,
- **symplectic group**, Sp(V), which preserves a symplectic form on V (a non-degenerate alternating form),
- **unitary group**, U(V), which, when F = **C**, preserves a non-degenerate hermitian form on V.

These groups provide important examples of Lie groups.

Related groups

Projective linear group

The projective linear group PGL(n, F) and the projective special linear group PSL(n,F) are the quotients of GL(n,F) and SL(n,F) by their centers (which consist of the multiples of the identity matrix therein); they are the induced action on the associated projective space.

Affine group

The affine group Aff(n,F) is an extension of GL(n,F) by the group of translations in F^n. It can be written as a semidirect product:

$$\text{Aff}(n, F) = \text{GL}(n, F) \ltimes F^n$$

where GL(n, F) acts on F^n in the natural manner. The affine group can be viewed as the group of all affine transformations of the affine space underlying the vector space F^n.

One has analogous constructions for other subgroups of the general linear group: for instance, the special affine group is the subgroup defined by the semidirect product, SL(n, F) $\ltimes$ F^n, and the Poincaré group is the affine group associated to the Lorentz group, O(1,3,F) $\ltimes$ F^n.

General semilinear group

The general semilinear group ΓL(*n*,*F*) is the group of all invertible semilinear transformations, and contains GL. A semilinear transformation is a transformation which is linear "up to a twist", meaning "up to a field automorphism under scalar multiplication". It can be written as a semidirect product:

ΓL(*n*, *F*) = Gal(*F*) ⋉ GL(*n*, *F*)

where Gal(*F*) is the Galois group of *F* (over its prime field), which acts on GL(*n*, *F*) by the Galois action on the entries.

The main interest of ΓL(*n*, *F*) is that the associated projective semilinear group PΓL(*n*, *F*) (which contains PGL(*n*, *F*)) is the collineation group of projective space, for *n* > 2, and thus semilinear maps are of interest in projective geometry.

Infinite general linear group

The **infinite general linear group** or **stable general linear group** is the direct limit of the inclusions $\mathrm{GL}(n, F) \to \mathrm{GL}(n+1, F)$ as the upper left block matrix. It is denoted by either $\mathrm{GL}(F)$ or $\mathrm{GL}(\infty, F)$, and can also be interpreted as invertible infinite matrices which differ from the identity matrix in only finitely many places.

It is used in algebraic K-theory to define K_1, and over the reals has a well-understood topology, thanks to Bott periodicity.

It should not be confused with the space of (bounded) invertible operators on a Hilbert space, which is a larger group, and topologically much simpler, namely contractible – see Kuiper's theorem.

Notes

[1]   Here rings are assumed to be associative and unital.
[2]   Since the Zariski topology is coarser than the metric topology; equivalently, polynomial maps are continuous.
[3]   A maximal compact subgroup is not unique, but is essentially unique, hence one often refers to "the" maximal compact subgroup.
[4]   Galois, Évariste (1846). "Lettre de Galois à M. Auguste Chevalier" (http://visualiseur.bnf.fr/ark:/12148/cb343487840/date1846). *Journal des mathématiques pures et appliquées* **XI**: 408–415. . Retrieved 2009-02-04, GL(*v*,*p*) discussed on p. 410.

External links

- "GL(2,p) and GL(3,3) Acting on Points" (http://demonstrations.wolfram.com/GL2PAndGL33ActingOnPoints/) by Ed Pegg, Jr., Wolfram Demonstrations Project, 2007.

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